Solutions to Selected Exercises

I am indebted to Krzysztof Smutek for having suggested to write this chapter. He worked out many of these solutions, and I am grateful for his permission to reproduce them here. There are indications on how to solve almost every exercise, except those which are embarrassingly easy, or those whose solutions can be found in the given references. Please be aware that I did not work out the exercises with the same dedication as I wrote the main text, and brace yourself for a higher density of typos and possibly a positive density of plain nonsense. After all, the point of the exercises is to make the reader (not the author) work.

Exercise 1.2.1 An operator A equals zero if and only if $(A(x), y) = 0$ for each $x, y \in \mathcal{H}$, so that two operators A and B are equal if and only if $(A(x), y) = (B(x), y)$ for all x, y. To prove (a) we write

$$(x, (\alpha A)(y)) = \alpha(x, A(y)) = \alpha(A^\dagger(x), y) = (\alpha^* A^\dagger(x), y) ,$$

using that the inner product is anti-linear in the first variable. To prove (b) we write

$$(x, (AB)(y)) = (x, A(B(y)) = (A^\dagger(x), B(y)) = (B^\dagger(A^\dagger(x), y) = ((B^\dagger A^\dagger)(x), y) .$$

Exercise 1.2.2 If $x \in F^\perp$ and $y \in F$ then $(A(x), y) = (x, A(y)) = 0$ since $A(y) \in F$. So $A(x) \in F^\perp$.

Exercise 1.2.3 The key here is that in a *complex* finite-dimensional Hilbert space an operator has at least one eigenvector. This is because we may assume that the space is $\mathbb{C}^n$ and then the characteristic polynomial $\det(A - \lambda 1)$ has at least one root. By Exercise 1.3.2, the orthogonal complement G of the corresponding eigenvector satisfies $A(G) \subset G$ and it suffices to apply the induction hypothesis to G.

Exercise 1.3.1 All you have to prove is that if $\Phi(\xi) = \int \mathrm{d}x f(x)\xi(x)$ for a nice function f then $\Phi'(\xi) := -\Phi(\xi') = \int \mathrm{d}x f'(x)\xi(x)$. This follows from integration by parts.

Exercise 1.4.1 (a) For example, you may deduce from the fact that $\int \mathrm{d}x \delta(x)\xi(x) =$

0 whenever $\xi(x) = 0$ that $\delta(x) = 0$ if $x \neq 0$. Then

$$\xi(0) = \int \mathrm{d}x\delta(x)\xi(x) = \int_{\{x=0\}} \mathrm{d}x\delta(x)\xi(x) = \xi(0)\int \mathrm{d}x\delta(x) ,$$

so that the δ function has integral 1. (Of course this makes no mathematical sense, but neither does the statement you try to "prove".)

(b) One should have $\delta(0) = \infty$, but, as my kindergarten teacher said, "infinity is not a number."

(c) You may argue that $\delta'(x) = 0$ if $x \neq 0$, but what is the value of $\delta'(0)$?

Exercise 1.4.2 Considering for example for any y the continuous function $\eta_x(y) = \xi(x, y)$, one has

$$\iint \mathrm{d}x\mathrm{d}y\xi(x,y)\delta(x-y) = \int \mathrm{d}x \int \mathrm{d}y\eta_x(y)\delta(x-y) = \int \mathrm{d}x\eta_x(x) = \int \mathrm{d}x\xi(x,x) .$$

Exercise 1.4.3 If ζ and η are test functions,

$$\begin{aligned}\int \mathrm{d}x\mathrm{d}y\zeta(x)\eta(y)\int \mathrm{d}z\delta(x-z)\delta(z-y)\xi(z) &= \int \mathrm{d}z\zeta(z)\eta(z)\xi(z)\\ &= \int \mathrm{d}x\mathrm{d}y\zeta(x)\eta(y)\delta(x-y)\xi(x) .\end{aligned}$$

Exercise 1.5.1 We start with the relation

$$\hat{f}(p) = \sqrt{2\pi}\mathcal{F}_m(f)(p/\hbar) .$$

Consider the function $\eta(p) = \hat{f}(p\hbar)$. Changing p into $p\hbar$ in the previous relation and applying $\mathcal{F}_m^{-1}$ to both sides, we get, using the formula for $\mathcal{F}_m^{-1}$,

$$f(x) = \frac{1}{\sqrt{2\pi}}\mathcal{F}_m^{-1}(\eta)(x) = \frac{1}{2\pi}\int \mathrm{d}y\eta(y)\exp(\mathrm{i}xy) .$$

Making the change of variable $y \to p/\hbar$ yields the desired relation.

Exercise 1.5.2 For example,

$$\check{\xi}(x_1,\ldots,x_n) = \int\cdots\int \frac{\mathrm{d}p_1}{2\pi\hbar}\cdots\frac{\mathrm{d}p_n}{2\pi\hbar}\exp\Big(\mathrm{i}\sum_{k\leq n} x_k p_k/\hbar\Big)\xi(p_1,\ldots,p_n) .$$

Exercise 2.1.1 In mathematical notation, if $x = |\alpha\rangle$ then this operator P_x is such that $P_x(y) = (x,y)x$. It is the orthogonal projection on $\mathbb{C}x$.

Exercise 2.2.1. Assume first that A and A' commute. Consider a basis $(|j\rangle)_{j\leq n}$ of eigenvectors for A, which are also eigenvectors for A' since A' commutes with A. Let (λ_i) and (λ'_i) be the corresponding eigenvalues. The probability of measuring a value of $\mathcal{O}$ equal to λ_i is then $|\langle i|\alpha\rangle|^2$. If we measure first $\mathcal{O}'$, the probability of measuring a value of λ'_j is $|\langle j|\alpha\rangle|^2$. After this measurement, the system is in state

$|j\rangle$ and the measurement of $\mathcal{O}$ will be λ_j. So we will measure λ_i only if $j = i$, and this occurs with the same probability $|\langle i|\alpha\rangle|^2$ as if we had not measured $\mathcal{O}'$ before we measured $\mathcal{O}$. Assume now that A and A' do not commute, so that there exists an eigenvector $|i\rangle$ of A which is not an eigenvector of A'. Assume that the system is in state $|i\rangle$. If we measure $\mathcal{O}$ first we are guaranteed to obtain the value λ_i. This is not the case if we measure $\mathcal{O}'$ first, because after this measurement the state of the system is an eigenvector of A', so is not of the type $\alpha|i\rangle$.

Exercise 2.5.4 We use integration by parts to write

$$(g, A(f)) = \mathrm{i}\int_0^1 \mathrm{d}x g(x)^* f'(x) = \mathrm{i}g(1)^* f(1) - \mathrm{i}g(0)^* f(0) - \mathrm{i}\int_0^1 \mathrm{d}x g'(x)^* f(x)$$

i.e. $(g, A(f)) = (A(g), f) + \mathrm{i}g(1)^* f(1) - \mathrm{i}g(0)^* f(0)$. Thus A is not symmetric on the domain $\mathcal{D}$ but it is symmetric on the domain $\mathcal{D}_\alpha$.

Exercise 2.5.7 Since A is symmetric $\mathcal{D}(A) \subset \mathcal{D}(A^\dagger)$. This implies that $\mathcal{D}((A^\dagger)^\dagger) \subset \mathcal{D}(A^\dagger)$. But since $A^\dagger$ is symmetric we have $\mathcal{D}(A^\dagger) \subset \mathcal{D}((A^\dagger)^\dagger)$ so that $\mathcal{D}(A^\dagger) = \mathcal{D}((A^\dagger)^\dagger)$ and A is self-adjoint.

Let us recall the following simple fact of functional analysis, which is useful to solve the next exercises: in a Hilbert space,

$$\sup_{\|y\| \le 1} |(x, y)| = \|x\| \ ,$$

as follows from the choice (when $x \neq 0$) of $y = x/\|x\|$. In particular if $|(x, y)| \le C\|y\|$ then $\|x\| \le C$.

Exercise 2.5.8 (a) It is straightforward that $\mathcal{D}(A)$ is a dense subspace on which A makes sense.
(b) For $x, y \in \mathcal{D}(A)$ we have $(x, A(y)) = \sum_{n\ge 0} x_n^* \lambda_n y_n$ whereas $(A(x), y) = \sum_{n\ge 0} \lambda_n^* x_n^* y_n$, from which the result follows.
(c) If $x \in \mathcal{D}(A^\dagger)$ then for all y in $\mathcal{D}(A)$,

$$|(x, A(y))| = \Big|\sum_{n\ge 0} x_n^* \lambda_n y_n\Big| = |(z, y)| \le C\|y\|$$

where $z = (\lambda_n^* x_n)_{n\ge 0}$. Thus $\|z\|^2 = \sum_{n\ge 0} |\lambda_n x_n|^2 \le C^2$ and $x \in \mathcal{D}(A)$. Furthermore since $(x, A(y)) = (z, y)$ we have $A^\dagger(x) = z$.

Exercise 2.5.9 The operator B is defined by $B(y) = (B_n(y_{n+1}))_{n\ge 0}$. For $x \in \mathcal{D}(A)$ and $y \in \mathcal{H}$ we have

$$(y, A(x)) = \sum_{n\ge 1} (y_n, A_{n-1}(x_{n-1})) = \sum_{n\ge 0} (B_n(y_{n+1}), x_n) \ .$$

This makes it obvious that $\mathcal{D}(B) \subset \mathcal{D}(A^\dagger)$. When $y \in \mathcal{D}(A^\dagger)$ we have $|(y, A(x))| \le C\|x\|$, and thus $\sum_{n\ge 0} \|B_n(y_{n+1})\|^2 \le C^2$ so that $y \in \mathcal{D}(B)$ and $B = A^\dagger$. The proof of the equality $A = B^\dagger$ is similar.

Exercise 2.5.10 Consider $y \in \mathcal{D}(A^\dagger)$. We have to prove that $y \in \mathcal{D}(A)$. Since $A - \mathrm{i}1$ is onto, we can find $x \in \mathcal{D}(A)$ such that $A^\dagger(y) - \mathrm{i}y = A(x) - \mathrm{i}x = A^\dagger(x) - \mathrm{i}x$, using Lemma 2.5.5 in the second equality. Thus $(A^\dagger - \mathrm{i}1)(y - x) = 0$. Hence for any z it holds $0 = ((A^\dagger - \mathrm{i}1)(x - y), z) = (x - y, (A + \mathrm{i}1)(z))$ and this implies that $x - y = 0$ since $A + \mathrm{i}1$ is onto. Thus $y = x \in \mathcal{D}(A)$.

Exercise 2.5.11 This is obvious. For example if $x = (x_n)_{n\geq 1} \in \mathcal{H}$ then $y = (x_n/(\lambda_n + \mathrm{i}))_{n\geq 1} \in \mathcal{H}$ and $(A + \mathrm{i}1)(y) = x$.

Exercise 2.5.12 Using Fourier series transports us to the space of square-integrable sequences $(x_n)_{n\in\mathbb{Z}}$, and transports A to an operator A' whose domain contains the sequences with finite support, and for such a sequence is defined as the transformation $(x_n) \mapsto (2\pi n x_n)$. Consequently by a previous argument a sequence (x_n) in the domain of $A^\dagger$ is such that $\sum_{n\in\mathbb{Z}} n^2 |x_n|^2 < \infty$. In particular $\sum_{n\in\mathbb{Z}} |x_n| < \infty$, so that the corresponding Fourier series converges absolutely and its sum is continuous.

Exercise 2.5.16 The operator T is bounded so that its graph is closed. The graph of its restriction to $\mathcal{L}_0$ is closed so that the graph of A is closed. For $f \in L^2$ and $g \in \mathcal{L}_0$ integration by parts and approximation by smooth functions shows that $(T(f), g) = -(f, T(g))$. This proves that A is symmetric. Moreover A is not self-adjoint because the previous formula implies that $T(L^2)$ is contained in the domain of $A^\dagger$ whereas $T(L^2)$ is larger than $T(\mathcal{L}_0)$.

Exercise 2.5.17 Denoting by A the "multiplication by x operator", the domain of $A^\dagger$ consists of the functions ψ such that $|(\psi, A(\varphi))| \leq C\|\varphi\|_2$ i.e.

$$\Big|\int \mathrm{d}x (x\psi(x)^*)\varphi(x)\Big| \leq C\|\varphi\|_2 \;,$$

so that $x\psi \in L^2$.

Exercise 2.5.18 A function φ is a eigenvector of eigenvalue a for the multiplication by x operator if $a\varphi = x\varphi$ in $L^2(\mathbb{R}^2, \mathrm{d}\mu, \mathbb{C})$, i.e. $(x - a)\varphi = 0$ μ-a.e. Thus $\varphi = 0$ μ-a.e. on $\mathbb{R} \setminus \{a\}$ and since φ is not zero we must have $\mu(\{a\}) \neq 0$.

Exercise 2.5.19 As in the previous exercise, the space of eigenvectors with eigenvalue a identifies to the space of functions $\varphi(x, y)$ which are zero for $x \neq a$, i.e. with $L^2(\mathrm{d}\mu')$ where $\mathrm{d}\mu'$ is the restriction of $\mathrm{d}\mu$ to $\{a\} \times \mathbb{R}$. It then suffices to prove that for a measure $\mathrm{d}\nu$ on $\mathbb{R}$ which gives finite measure to bounded sets the space $L^2(\mathrm{d}\nu)$ has dimension n if and only if $\mathrm{d}\nu$ is carried by n points but not by $n - 1$ points. If $L^2(\mathrm{d}\nu)$ is finite-dimensional, $\mathrm{d}\nu$ cannot have a continuous part, and can charge only finitely many points. It is straightforward to check that its dimension is exactly the number n of points it charges.

Exercise 2.5.20 We write, using mathematical notation for clarity,

$$\langle\gamma|\alpha\rangle = (|\gamma\rangle, |\alpha\rangle) = (|\alpha\rangle, |\gamma\rangle)^* = (|\alpha\rangle, A|\beta\rangle)^* = (A^\dagger(|\alpha\rangle), |\beta\rangle)^* = (|\beta\rangle, A^\dagger(|\alpha\rangle)) \;,$$

i.e. $\langle\gamma|\alpha\rangle = \langle\beta|A^\dagger|\alpha\rangle$ so that $\langle\gamma| = \langle\beta|A^\dagger$.

Exercise 2.5.21 It is immediate that the domain of P is invariant under T and that for f in this domain we have $PT(f) = TP(f)$. If another observer, Alice, is located at a fixed position a, a point of coordinate x in Alice's reference frame has coordinate $x + a$ in my frame. When I describe a particle by the position state space function f, the quantity $|f(x)|^2$ describes the probability density of finding the particle at location x. The location x has coordinate $x - a$ in Alice's reference frame. It seems then most reasonable to assume that Alice will describe the same particle by the position state function $T(f)$ given by $T(f)(x) = f(x + a)$. Since T commutes with P, the momentum operator P produces the same value when applied to these two state functions,

$$(T(f), PT(f)) = (T(f), TP(f)) = (f, Pf) .$$

This means that Alice measures the same momentum of the particle as I do, as is confirmed by experience.

Exercise 2.5.22 The induction is straightforward. Using that

$$\|AB^n\| \le \|A\|\|B\|\|B^{n-1}\|$$

and similarly for $\|B^n A\|$ we obtain

$$n\|B^{n-1}\| = \|AB^n - B^n A\| \le 2\|A\|\|B\|\|B^{n-1}\| ,$$

which cannot hold for large n.

Exercise 2.7.1 Recalling the Fourier transform U from $\mathcal{H}$ to $\mathcal{H}'$, let $T' := UTU^{-1}$. It is straightforward that for a test function ξ we have $\check{\xi}(a+x) = \check{\eta}(x)$ where $\eta(p) = \exp(\mathrm{i}ap/\hbar)\xi(p)$. This means that T' is the operator "multiplication by $\exp(\mathrm{i}ap/\hbar)$", and this operator commutes with the operators "multiplying with a function of p".

Exercise 2.8.1 One must not be shy in this type of formal manipulations, and one writes

$$\int \mathrm{d}y\psi(y) \int \mathrm{d}z\delta_y(z)\delta_x(z) = \int \mathrm{d}z\delta_x(z) \int \mathrm{d}y\psi(y)\delta_y(z) , \tag{P.6}$$

and then one uses that

$$\int \mathrm{d}y\psi(y)\delta(y - z) = \psi(z) . \tag{P.7}$$

The quantity (P.6) then equals $\int \mathrm{d}z\delta_x(z)\psi(z)$, which is what one wanted to show.

Exercise 2.8.2 There is a complete symmetry between position and momentum state space, so you may transpose any of the previous arguments you wish. For example, since $|p\rangle$ is supposed to stand for a state of given momentum, and since in momentum state space the momentum operator is the "multiplication by p operator", then $|p\rangle$ has to correspond to a function which is not zero only at p, i.e. a multiple of δ_p. Is has to correspond precisely to $2\pi\hbar\delta_p$ since for the natural measure on momentum space this function is of integral 1, which is required to make formulas such as the equivalent of (2.26) work. Another approach is that by analogy with (2.26) if the test function ξ is seen as an element $|\xi\rangle$ of position state space, one should have $|\xi\rangle = \int (\mathrm{d}p/(2\pi\hbar))\xi(p)|p\rangle$. Integration in p of the relation $\langle p|p'\rangle\xi(p) = 2\pi\hbar\delta(p-p')\xi(p)$ then yields as required $\langle\xi|p'\rangle = \xi(p')$.

Exercise 2.9.1 This might be obvious at the formal level, but if you try to think about it, you will need to write at least a line.

Exercise 2.10.2 On the one hand

$$U(abc) = U((ab)c) = r(ab,c)U(ab)U(c) = r(ab,c)r(a,b)U(a)U(b)U(c)$$

and on the other hand

$$U(abc) = U(a(bc)) = r(a,bc)U(a)U(bc) = r(a,bc)r(b,c)U(a)U(b)U(c)\ .$$

Exercise 2.14.1 Write $\widehat{V(t)(f)}(p) = \int \mathrm{d}x \exp(-\mathrm{i}xp/\hbar)f(x+t)$ and make the change of variables $y = x+t$.

Exercise 2.14.3 (a) The only point which is not obvious is strong continuity. If h is uniformly bounded by B on the support of f one simply use that $\|(\exp(\mathrm{i}th/\hbar)-1)f\|_2 \le B|t|\|f\|_2/\hbar$ so that then $\lim_{t\to 0}\|(\exp(\mathrm{i}th/\hbar)-1)f\|_2 = 0$ and the result since the set of f where this limit is 0 is closed in norm. (b) If f belongs to the domain of A, the function $\theta(f,t) := (U(t)(f) - U(0)(f))/(t\hbar)$ remains bounded in L^2 as $t \to 0$. Since this function converges point-wise to hf we have $hf \in L^2$. Conversely if $hf \in L^2$, then $\|\theta(f,t)\|_2 \le \|hf\|_2/\hbar$ so that by approximation one reduces to the case where h is bounded on the support of f to prove that $f \in \mathcal{D}$ and $A(f) = hf$. (c) is a consequence of (a) and (b).

Exercise 2.17.1 One has to show first that $|(y, a(x))| \le C\|x\|$ for $x \in \mathcal{D}$ if and only if $y \in \mathcal{D}$. The "if" part is easy and the "only if part" is done by taking $x_{n+1} = \sqrt{n}y_n$ for $n \le k$ and $x_{n+1} = 0$ otherwise so that

$$(y, a(x)) = \sum_{n\le k}\sqrt{n(n+1)}|y_n|^2 \le C\|x\| = C\sqrt{\sum_{n\le k} n|y_n|^2}$$

and thus $y \in \mathcal{D}$. The computation of $a^\dagger$ is then straightforward.

Exercise 2.17.2 The vector $\sum_{n\ge 0}(\lambda^n/\sqrt{n!})e_n$ has this property.

Exercise 2.18.1 This has basically been done in Exercise 2.5.8.

Exercise 2.18.3 Consider the Taylor polynomial $P_n(x) = \sum_{k\le n}(\mathrm{i}tx)^k/k!$. It holds $|P_n(x) - \exp(\mathrm{i}tx)| = |\sum_{k\ge n+1}(\mathrm{i}tx)^k/k!| \le \exp|tx|$. Since $|P_n(x) - \exp(\mathrm{i}tx)|$ goes pointwise to zero and since $\exp|tx|\varphi_0$ is square-integrable, Lebesgue's convergence theorem implies that $P_n(x)\varphi_0(x) \to \exp(\mathrm{i}tx)\varphi_0(x)$ in L^2.

Exercise 2.18.4 Compute $a(t)(e_{n+1}) = \exp(\mathrm{i}tH/\hbar)a(\exp(-\mathrm{i}tH/\hbar)(e_{n+1}))$ $= \exp(\mathrm{i}H/\hbar)a(\exp(-\mathrm{i}t(n+3/2)\omega)e_{n+1}) = \sqrt{n+1}\exp(-\mathrm{i}t(n+3/2)\omega)\exp(\mathrm{i}tH/\hbar)e_n$ $= \sqrt{n+1}\exp(-\mathrm{i}t\omega)e_n = \exp(-\mathrm{i}t\omega)a(e_{n+1})$.

Exercise 3.1.1 $\dim \mathcal{H}_1 \otimes \mathcal{H}_2 = \dim \mathcal{H}_1 \times \dim \mathcal{H}_2$ whereas $\dim(\mathcal{H}_1 \times \mathcal{H}_2) = \dim \mathcal{H}_1 + \dim \mathcal{H}_2$.

Exercise 3.1.2 Show first that to prove that a unitary group $U(t)$ is strongly continuous it suffices to prove that $t \mapsto U(t)(y)$ is continuous at $t = 0$ for each y is a set large enough that its closed linear span is $\mathcal{H}$. Taking $n = 2$ it suffices then to consider the case where $y = x_1 \otimes x_2$. We write

$$U(t)(x_1 \otimes x_2) - x_1 \otimes x_2 = (U_1(t)(x_1) - x_1) \otimes x_2 + U_1(t)(x_1) \otimes (U_2(t)(x_2) - x_2) ,$$

so that, using that $U_1(t)$ and $U_2(t)$ are unitary,

$$\|U(t)(x_1 \otimes x_2) - x_1 \otimes x_2\| \leq \|U_1(t)(x_1) - x_1\|\|x_2\| + \|x_1\|\|U_2(t)(x_2) - x_2\| .$$

Assuming now that each x_k belongs to the domain of A_k we write $U_k(t)(x_k) = x_k + \mathrm{i}t\hbar A_k(x_k) + o(t)$ where $o(t)/t$ goes to 0 with t, we expand the product $U_1(t)(x_1) \otimes \ldots \otimes U_n(t)(x_n)$ and we look at the terms of order t to obtain (3.5).

Exercise 3.2.2 The one thing which is not completely obvious is that these elements span $\mathcal{H}_{2,s}$. To see this, we recall that the elements of the type $x \otimes y$ span $\mathcal{H} \otimes \mathcal{H}$. Thus every element z of $\mathcal{H} \otimes \mathcal{H}$ may be approximated by a linear combination of the $x \otimes y$. If z is moreover symmetric, it is also approximated by the same linear combination of the $y \otimes x$, and hence by a linear combination of tensors of the form $x \otimes y + y \otimes x$, and the conclusion should then be obvious.

Exercise 3.2.3 Orthonormality of this sequence follows easily from (3.8) and the definition of inner product on $\mathcal{H}_{n,s}$, see (3.2). To show that it indeed spans the whole space one can simply use the same arguments as in the solution to Exercise 3.2.2.

Exercise 3.3.1 We have to prove that for $\alpha \in \mathcal{H}_{n,s}$ and $\beta \in \mathcal{H}_{n+1,s}$ it holds $(A^\dagger(\gamma)(\alpha), \beta) = (\alpha, A(\gamma), \beta)$. It suffices to prove these formulas where γ, α, β are basis elements, $\gamma = e_k$, $\alpha = |n_1, n_2, \ldots, n_k, \ldots\rangle, \beta = |n'_1, n'_2, \ldots, n'_k, \ldots\rangle$. In that case $(A^\dagger(\gamma)(\alpha), \beta) = (\alpha, A(\gamma), \beta) = 0$ unless $n'_j = n_j$ for $j \neq k$ and $n'_k = n_k + 1$, in which case $(A^\dagger(\gamma)(\alpha), \beta) = (\alpha, A(\gamma), \beta) = \sqrt{n_k + 1}$.

Exercise 3.3.3 Given an orthonormal basis (e_i) of $\mathcal{H}$, for integers $i_1, \ldots, i_n$ we identify the tensor $e_{i_1} \otimes \ldots \otimes e_{i_n}$ with the function $e_{i_1,\ldots i_n}$ on $\mathbb{R}^n$ given by $e_{i_1,\ldots i_n}(x_1, \ldots, x_n) = \prod_{j \leq n} e_{n_j}(x_j)$. These functions form an orthonormal system, and in this way we obviously construct an isometry for $\mathcal{H}_{n,s}$ into a subspace of the symmetric functions on $\mathbb{R}^n$. To prove that is is onto, it suffices to prove that the functions $e_{i_1,\ldots,i_n}$ span $L^2(\mathbb{R}^n)$ (so that they form an orthonormal basis of $L^2(\mathbb{R}^n)$) and to use an expansion of a symmetric function of $L^2(\mathbb{R}^n)$ on this basis. Using induction over n, this reduces to showing that a function of $L^2(\mathbb{R}^n)$ which is orthogonal to every function of the type $u(x_n)v(x_1, \ldots, x_{n-1})$ is zero, an easy exercise of measure theory. Recall that elements of the type (3.21) span $\mathcal{H}_{n,s}$ (this can be proved as Exercise 3.2.3). Thus it suffices to prove these formulas when f is of this type, and then they quickly reduce to (3.22) and (3.23). The last statement follows by combining (3.24) and (3.25).

Exercise 3.4.3 The domains of $A(\xi)$ and $A^\dagger(\eta)$ are respectively given by

$$\mathcal{D}(A(\xi)) = \Big\{(\alpha(n))_{n\geq 1} \in \mathcal{B} \; ; \; \sum_{n\geq 0} \|A_n(\xi)(\alpha(n))\|^2 < \infty\Big\}$$

$$\mathcal{D}(A^\dagger(\eta)) = \Big\{(\alpha(n))_{n\geq 0} \in \mathcal{B} \; ; \; \sum_{n\geq 0} \|A_n^\dagger(\eta)(\alpha(n))\|^2 < \infty\Big\} \, ,$$

and the operators $A(\gamma)$ and $A^\dagger(\gamma)$ are adjoint to each other.

Exercise 3.4.4 This requires simply care and patience. For example, for $\alpha \in \mathcal{H}_{n,s}$,

$$A(\xi)A^\dagger(\eta)(\alpha)_{i_1,\ldots,i_n} = \sum_{i\geq 1, \ell\leq n} \xi_i^* \eta_{i_\ell} \alpha_{i_1,\ldots,\hat{i}_\ell,\ldots,i_n,i} + \alpha_{i_1,\ldots,i_n} \sum_{i\geq 1} \xi_i^* \eta_i \, ,$$

whereas the computation of $A^\dagger(\eta)A(\xi)(\alpha)_{i_1,\ldots,i_n}$ yields only the first summation to the right and this proves (3.33).

Exercise 3.5.1 This is obvious from (3.39).

Exercise 3.6.1 One just performs the same computation inside every eigenspace.

Exercise 3.7.1 Consider the case where $\xi(x,y) = h(x)g(y)$ where $h, g \in \mathcal{S}$. Then $S = A^\dagger(h)A(g^*)$ and the desired formula is the last assertion of Exercise 3.3.3. Approximating $\xi \in \mathcal{S}_2$ by a sum of functions of the preceding type concludes the argument.

Exercise 3.7.2 *First solution.* We start with

$$a(y)a(x)(f)(x_1, \ldots, x_{n-2}) = \sqrt{n(n-1)} f(x_1, \ldots, x_{n-2}, x, y) \, ,$$

and as in (3.52) we obtain

$$\begin{aligned} a^\dagger(x)a(y)a(x)(f)(x_1, \ldots, x_{n-1}) &= \sqrt{n} \sum_{k\leq n-1} \delta_x(x_k) f(x_1, \ldots, x_{n-1}, y) \\ &=: \sqrt{n} g(x_1, \ldots, x_{n-1}) \, , \end{aligned}$$

so that

$$a^\dagger(y)a^\dagger(x)a(y)a(x)(f)(x_1,\ldots,x_n) = \sum_{\ell\le n}\delta_y(x_\ell)g(x_1,\ldots,\widehat{x_\ell},\ldots,x_n)$$

$$= \sum_{\ell\le n}\delta_y(x_\ell)\Big(\delta_x(x_1)+\ldots+\widehat{\delta_x(x_\ell)}+\ldots+\delta_x(x_n)\Big)f(x_1,\ldots,\widehat{x_\ell},\ldots,x_n,y)$$

$$= \sum_{\ell\ne k}\delta_y(x_\ell)\delta_x(x_k)f(x_1,\ldots,\widehat{x_\ell},\ldots,x_n,y) = \sum_{k\ne\ell}\delta_y(x_\ell)\delta_x(x_k)f(x_1,\ldots,x_n)\ .$$

One then finishes with the relation $\int \mathrm{d}x\mathrm{d}yV(x,y)\delta_y(x_\ell)\delta_x(x_k) = V(x_k,x_\ell)$.
Second solution. This solution is more formal. Using the relation $a^\dagger(y)a(y) = a(y)a^\dagger(x) - \delta(y-x)1$ and assuming that $V(x,y) = V_1(x)V_2(y)$,

$$H_V = \int \mathrm{d}xV_1(x)a^\dagger(x)a(x)\int \mathrm{d}yV_2(y)a^\dagger(y)a(y) - \int \mathrm{d}xV(x,x)a^\dagger(x)a(x)\ .$$

One then uses the formula (3.53). The case of a general function $V(x,y)$ is recovered by approximation.

Exercise 3.7.3 The sensible way is to require that when $V(x,y) = \xi(x)\eta(y)$ this is $A^\dagger(\xi)A(\eta^*)$. For $f \in \mathcal{H}_{n,s}$, $\iint \mathrm{d}x\mathrm{d}yV(x,y)a^\dagger(x)a(y)f$ is then the symmetrized of the function $\int \mathrm{d}yV(x_1,y)f(y,x_2,\ldots,x_n)$.

Exercise 3.7.4 As explained just before, the first term is the sum of the kinetic energy of the individual particles, and the second term means that any two different particles located at x and y interact with a potential $V(x,y)$.

Exercise 3.8.1 Recalling that $(\xi,\eta) = [A'(\xi),A'^\dagger(\eta)]$ one compares the formulas

$$(\xi,\eta) = \iint \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi\hbar)^3}\xi(\boldsymbol{p})^*\eta(\boldsymbol{p}')(2\pi\hbar)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$$

and

$$[A'(\xi),A'^\dagger(\eta)] = \Big[\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3}\xi(\boldsymbol{p})^*a(\boldsymbol{p}),\int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi\hbar)^3}\eta(\boldsymbol{p}')a^\dagger(\boldsymbol{p}')\Big]$$

$$= \iint \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi\hbar)^3}\xi(\boldsymbol{p})^*\eta(\boldsymbol{p}')[a(\boldsymbol{p}),a^\dagger(\boldsymbol{p}')]\ . \quad \text{(P.8)}$$

Exercise 3.8.2 The quantity $b^\dagger(\boldsymbol{p})|0\rangle$ represents the ideal case of a single particle state having exactly momentum $\boldsymbol{p}$, whereas $b^\dagger(\boldsymbol{p})b^\dagger(\boldsymbol{p}')|0\rangle$ represents (the ideal case of) a two-particle state, the particles having momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$.

Exercise 3.8.3 The meaning of (3.60) is that we have $[\int \mathrm{d}^3\boldsymbol{p}/(2\pi\hbar)^3\xi(\boldsymbol{p})b(\boldsymbol{p}),\int \mathrm{d}^3\boldsymbol{p}/(2\pi\hbar)^3\xi'(\boldsymbol{p})b^\dagger(\boldsymbol{p})] = \int \mathrm{d}^3\boldsymbol{p}/(2\pi\hbar)^3\xi(\boldsymbol{p})^*\xi(\boldsymbol{p})$. The point is that the factor $\mathrm{d}^3\boldsymbol{p}/(2\pi\hbar)^3$ has dimension $[l^{-3}]$ because $\hbar$ has the dimension of an action, the product of a momentum by a length, and the natural way to make the dimensions equal in both sides of the previous equality is to think of $b(\boldsymbol{p})$ and $b^\dagger(\boldsymbol{p})$ as having dimension $[l^{3/2}]$.

Exercise 3.9.1 Since (f_n) is an orthonormal basis, any square-integrable function g has an expansion $g = \sum_{n\geq 1} a_n f_n$ with $a_n = \int f_n^*(x)g(x)\mathrm{d}x$. Thus, formally

$$\int \mathrm{d}x\Big(\sum_{n\geq 1} f_n(x)^* f_n(y)\Big)g(x) = \sum_{n\geq 1} f_n(y)\int \mathrm{d}x f_n(x)^* g(x) = \sum_{n\geq 1} a_n f_n(y) = g(y) \ ,$$

and using (P.7) this is what one wanted to prove. Considering $h = \sum_{n\geq 1} b_n f_n$ you may also write

$$\lim_{k\to\infty} \iint \mathrm{d}x\mathrm{d}y\Big(\sum_{n\leq k} f_n(x)^* f_n(y)\Big)g(x)h(y)^* = \sum_{n\geq 1} b_n^* a_n = \int \mathrm{d}x h(x)^* g(x) \ ,$$

and this means that as a distribution of two variables, $\sum_{n\geq 1} f_n(x)^* f_n(y)$ is well-defined with the appropriate value $\delta(x-y)$.

Exercise 3.10.2 The only sensible definition of $\sum_k a_k(c(k)+c^\dagger(k))$ would be such that

$$\sum_k \alpha_k(c(k)+c^\dagger(k))(e_\emptyset) = \sum_k \alpha_k e_k \ ,$$

but the right-hand side is not defined when $\sum_{k\geq 1}|\alpha_k|^2 = \infty$. On the other hand, if $\sum_k |\alpha_k|^2 < \infty$ it is quite straightforward to check that the series $\sum_k \alpha_k(c(k) + c^\dagger(k))(x)$ converges when $x = c^\dagger(i_1)\cdots c^\dagger(i_n)(e_\emptyset)$.

Exercise 3.10.3 Consider a test function ξ and $a_k = \int \mathrm{d}x\xi(x)g_k(x)$. Using integration by parts one obtains that the sequence (a_k) decreases fast enough that there is no problem of convergence in what follows. First

$$\int \mathrm{d}x\xi(x)\frac{\partial^2\varphi(t,x)}{\partial t^2} = \gamma\sum_{k\in\mathcal{K}}\frac{a_k}{\sqrt{\omega_k}}\frac{\partial^2}{\partial t^2}(c(t,k)+c^\dagger(t,k)) \ . \tag{P.9}$$

Furthermore, using integration by parts

$$\int \mathrm{d}x\xi''(x)g_k(x) = \int \mathrm{d}x\xi(x)g_k''(x) = -\frac{k^2}{\hbar^2}\int \mathrm{d}x\xi(x)g_k(x) \ ,$$

so that

$$\int \mathrm{d}x\xi''(x)\varphi(t,x) = \gamma\sum_{k\in\mathcal{K}}\frac{a_k}{\sqrt{\omega_k}}\Big(-\frac{k^2}{\hbar^2}\Big)(c(t,k)+c^\dagger(t,k)) \ . \tag{P.10}$$

Since

$$\frac{\partial^2}{\partial t^2}(c(k,t)+c^\dagger(k,t)) = -\omega_k^2(c(k,t)+c^\dagger(k,t) \ ,$$

the required identity

$$\int \mathrm{d}x\Big(\frac{\partial^2}{\partial t^2}\varphi(x,t) - \alpha^2\frac{\partial^2}{\partial x^2}\varphi(x,t) + \beta\varphi(x,t)\Big)\xi(x) = 0$$

then simply results for (3.78), (P.9), (P.10) and the relation $\omega_k^2 = \alpha^2 k^2/\hbar^2 + \beta$.

Exercise 3.10.4 We write

$$2a(k)a^\dagger(-k) = (c(k) + \mathrm{i}c(-k))(c^\dagger(k) - \mathrm{i}c^\dagger(-k))$$

$$2a^\dagger(-k)a(k) = (c^\dagger(k) - \mathrm{i}c^\dagger(-k))(c(k) + \mathrm{i}c(-k)) ,$$

and the required relation follows from the fact that $c(k)c^\dagger(k) + c(-k)c^\dagger(-k) = c^\dagger(k)c(k) + c^\dagger(-k)c(-k)$.

Exercise 4.1.2 Reduce to the case where $A(e_0) = e_0$ by replacing A by LA where $L \in SO^\uparrow(1,3)$ is such that $L(A(e_0))$ is a multiple of e_0 and then multiplying A by a constant. If $\boldsymbol{u}$ is a unit vector of $\mathbb{R}^3$ and $y = (0, \boldsymbol{u})$ then $(e_0 \pm y, e_0 \pm y) = 0$ so that $(e_0 \pm Ay, e_0 \pm Ay) = 0$ and thus $(e_0, Ay) = 0$ and $1 = (Ay, Ay)$. That is, A fixes the span of e_1, e_2, e_3 and is an isometry on this span, so it is a Lorentz transformation since it fixes e_0.

Exercise 4.1.5. We write $x^\nu = \eta^{\nu\lambda}x_\lambda$ so that $L_{\mu\nu}x^\nu = L_{\mu\nu}\eta^{\nu\lambda}x_\lambda = L_\mu{}^\lambda x_\lambda$.

Exercise 4.1.6 Starting with $L_{\nu\mu}L^\nu{}_{\lambda'} = \eta_{\mu\lambda'}$ we obtain $\eta^{\lambda\mu}L_{\nu\mu}L^\nu{}_{\lambda'} = \eta^{\lambda\mu}\eta_{\mu\lambda'}$ which is the required equality.

Exercise 4.3.2 We may take for B the boost B_r given by the formula (4.22), where $\boldsymbol{r} = -\alpha\boldsymbol{x}, r^0 = \alpha\boldsymbol{x}^2/x^0$, where α is chosen so that $r^2 = (r^0)^2 - \boldsymbol{r}^2 = -\alpha^2\boldsymbol{x}^2x^2/(x^0)^2 = 1$.

Exercise 4.3.4 If R and S are two rotations which transform $B^s(e_0)$ into r we have to show that $RB^sR^{-1} = SB^sS^{-1}$, or equivalently, that $(S^{-1}R)B^s(S^{-1}R)^{-1} = B^s$. Now, $S^{-1}R$ is a rotation which fixes $B^s(e_0)$, hence it fixes e_3 (since it fixes e_0 because it is rotation, and since $B^s(e_0)$ is a linear combination of e_0 and e_3). Thus it suffices to show that such a rotation commutes with B_s. This is rather obvious if one writes the matrices of these transformations. To prove (4.23), we simply say that $B_r = RB^sR^{-1}$ where R transforms $B^s(e_0)$ into r so that $SB_rS^{-1} = (SR)B^s(SR)^{-1}$ where SR is a rotation which transforms $B^s(e_0)$ into $S(r)$. Thus this transformation equals $B_{S(r)}$.

Exercise 4.4.1 The pure boost B^s sends $(1,0,0,1)$ to $(\exp s, 0, 0, \exp s)$, and a suitable rotation sends this point to any point of the type $(\exp s, \boldsymbol{p})$ where $|\boldsymbol{p}| = \exp s$.

Exercise 4.4.2 In that domain X_m is almost flat. When f is non-zero only when $|\boldsymbol{p}|$ is much smaller than cm the formula (4.36) gives

$$\int_{X_m} \mathrm{d}\lambda_m(p)f(p) \simeq \frac{1}{mc}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3}f(p) . \tag{P.11}$$

Thus $\mathrm{d}\lambda_m$ is nearly proportional to the volume measure, although the proportionality factor is not very appealing.

Exercise 4.4.4 Let $\eta(p') = (2\pi\hbar)^3\sqrt{2\omega_{\boldsymbol{p}}}J(\delta_{\boldsymbol{p}}^{(3)})(p') = (2\pi\hbar)^3\sqrt{2\omega_{\boldsymbol{p}}}\sqrt{2\omega_{\boldsymbol{p}'}}\delta_{\boldsymbol{p}}^{(3)}(\boldsymbol{p}')$ so that

$$\int \mathrm{d}\lambda_m(p')\eta(p')\xi(p') = \int \frac{\mathrm{d}^3\boldsymbol{p}'}{2\omega_{\boldsymbol{p}'}}\sqrt{2\omega_{\boldsymbol{p}}}\sqrt{2\omega_{\boldsymbol{p}'}}\delta_{\boldsymbol{p}}^{(3)}(\boldsymbol{p}')\xi(p') = \xi(p) \ ,$$

which proves the claim.

Exercise 4.4.5 This is just a way to interpret (4.41).

Exercise 4.5.2 Assume first the U and U' are unitarily equivalent, and consider a unitary map W as in (4.44). Consider an orthonormal basis $(e_i)_{i\leq n}$ of $\mathcal{H}$, so that $(W(e_i))$ is an orthonormal basis of $\mathcal{H}'$, and $(W(e_j), U'(a)W(e_i)) = (e_j, W^{-1}U'(a)W(e_i)) = (e_j, U(a)(e_i))$, so that the matrix of $U'(a)$ in the base $(W(e_i))$ is the same as the matrix of $U(a)$ in the basis (e_i). The reverse direction is just as obvious, by using the operator W which sends the basis of $\mathcal{H}$ to the basis of $\mathcal{H}'$.

Exercise 4.5.9 If a representation is not irreducible there exists a non-trivial invariant subspace $\mathcal{G}$. We show that the orthogonal projector P on $\mathcal{G}$ commutes with all the operators $U(a)$. We note that since U is unitary the orthogonal complement $\mathcal{G}^\perp$ of $\mathcal{G}$ is also an invariant subspace. Then, writing an element $x \in \mathcal{H}$ as $x_1 + x_2$ where $x_1 \in \mathcal{G}$ and $x_2 \in \mathcal{G}^\perp$ then $U(a)(x) = U(a)(x_1)+U(a)(x_2)$ where $U(a)(x_1) \in \mathcal{G}$ and $U(a)(x_2) \in \mathcal{G}^\perp$ so that $PU(a)(x) = U(a)(x_1) = U(a)P(x)$.

Exercise 4.6.1 If $g(x) = a \cdot f(a^{-1} \cdot x)$ then $U(b)g(x) = b \cdot g(b^{-1} \cdot x) = ba \cdot f(a^{-1} \cdot (b^{-1} \cdot x)) = U(ba)(f)(x)$.

Exercise 4.7.1 Just compute $(-A^{-1}(a), A^{-1})(a, A) = (-A^{-1}(a) + A^{-1}(a), 1)$.

Exercise 4.8.3 (a) The formula $\int \mathrm{d}t\mathrm{d}x\varphi(x)\psi(x+t) = \int \mathrm{d}x\varphi(x)\int \mathrm{d}t\psi(t)$ follows by integrating in t first and using that $\int \mathrm{d}t\psi(x+t) = \int \mathrm{d}t\psi(t)$. Thus if for each t $\varphi(x)\psi(x+t)$ is zero a.e. then either $\int \mathrm{d}x\varphi(x) = 0$ or $\int \mathrm{d}t\psi(t) = 0$, and the conclusion since $\varphi, \psi \geq 0$. (b) Using left-invariance for the function $\theta(R) = \varphi(Rx)$ shows that $\int \mathrm{d}R\varphi(Rx) = \int \mathrm{d}\varphi(RTx)$ so that $a := \int \mathrm{d}R\varphi(Rx)$ is independent of x. Integrating in x and using that $\int \mathrm{d}\mu(x)\varphi(Rx) = \int \mathrm{d}\mu(x)\varphi(x)$ for any R yields the result. (c) is immediate by integrating in R first and using (b).

Exercise 4.8.5 Consider the map $p \to \beta p$ from X_m to $X_{\beta m}$. The image of $\mathrm{d}\lambda_m$ under this map is obviously invariant under the action of $SO^\uparrow(1,3)$ so that it is proportional to $\mathrm{d}\lambda_{\beta m}$. That is, there exists a number $B(=B(\beta))$ such that for each φ in $L^2(X_{\beta m}, \mathrm{d}\lambda_{\beta m})$ we have $\int \varphi(\beta p)^2\mathrm{d}\lambda_m(p) = B^2\int \varphi(p')^2\mathrm{d}\lambda_{\beta m}(p')$. The map $V : L^2(X_{\beta m}, \mathrm{d}\lambda_{\beta m}) \to L^2(X_m, \mathrm{d}\lambda_m)$ given by $V(\varphi)(p) = B^{-1}\varphi(\beta p)$ is then an isometry and it is straightforward to see that $U(a, A)V(\varphi)(p) = \exp(\mathrm{i}\beta\alpha(a, p))V(\varphi)(A^{-1}(p))$.

Exercise 4.10.2 Look at (4.69). If you think of $|p\rangle$ as a function in $L^2(X_m, \mathrm{d}\lambda_m)$, it is indeed $\delta_{m,p}$ which has this very property.

Exercise 4.10.3 For example the heuristic formula $\langle p'|\xi\rangle = \xi(p')$ is equivalent to $\int \mathrm{d}\lambda_m(p)\xi(p)\langle p'|p\rangle = \int \mathrm{d}\lambda_m(p)\xi(p)\delta_{m,p'}(p)$.

Exercise 5.1.2 This is quite obvious. Formally, taking the adjoint of (5.1) gets $\sqrt{c}\varphi(f)^\dagger = A^\dagger(\hat{f}) + A(\widehat{f^*}) = \sqrt{c}\varphi(f^*)$.

Exercise 5.1.8 We prove by induction over n that $T^n(f)(u) = \sum_{0\le k\le n} g_{n,k}(u)f^{(k)}(u)$ where $g_{n,k}(u) = P_{n,k}(\sinh u, \cosh u)/\cosh u^{n+1}$ where $P_{n,k}$ is polynomial in two variables of degree n. In particular the functions $|g_{n,k}|$ are integrable. This proves (a). For (b) it suffices (appealing to dominated convergence) to show that the functions $(1 - f_{\varepsilon,r})$ and $f^{(k)}_{\varepsilon,r}$ for $k = 1, 2, 3$ are bounded uniformly over r and converge pointwise to zero as $\varepsilon \to 0$. This is done by elementary bounds. For example, $|f'_{\varepsilon,r}(u)| \le a\exp(-a)$ where $a = \varepsilon\sqrt{r^2 + m^2c^2}\cosh(u + \tau)$.

Exercise 5.1.9 We denote by $d, d' \dots$ numerical constants, (which need not be the same at each occurrence). We set $\omega_r = \sqrt{r^2 + m^2c^2}$. Assuming again $\boldsymbol{x}$ to be in the z direction we now integrate in spherical coordinates to obtain

$$\begin{aligned} I_\varepsilon(x) &= d\int_0^\infty \mathrm{d}r \frac{r}{|\boldsymbol{x}|\omega_r}\exp(-\mathrm{i}\omega_r x^0/\hbar - \varepsilon\omega_r)\sin(|\boldsymbol{x}|r/\hbar) \\ &= \frac{d}{2}\int \mathrm{d}r \frac{r}{|\boldsymbol{x}|\omega_r}\exp(-\mathrm{i}\omega_r x^0/\hbar - \varepsilon\omega_r)\sin(|\boldsymbol{x}|r/\hbar) \\ &= d'\int \mathrm{d}r \frac{1}{x_0 + d''\varepsilon}\exp(-\mathrm{i}x^0\omega_r x^0/\hbar - \varepsilon\omega_r)\cos(|\boldsymbol{x}|r/\hbar)\ , \end{aligned} \tag{P.12}$$

where in the third inequality we have integrated by parts. More integration by parts show that the limit

$$I(x) = d\int \mathrm{d}r \exp(-\mathrm{i}\omega_r x^0/\hbar)\cos(|\boldsymbol{x}|r/\hbar)$$

exists. We further compute it by setting $r = mc\sinh u$, $x^0 = b\cosh\tau$, $|\boldsymbol{x}| = b\sinh\tau$ where b satisfies $b^2 = x^2$ and $\operatorname{sign} b = \operatorname{sign} x^0$. We then obtain (with $a = mcb/\hbar$)

$$I(x) = d\int \mathrm{d}u \frac{1}{\cosh\tau}\sinh u\big(\exp(-a\mathrm{i}\cosh(u-\tau) + \exp(-a\mathrm{i}\cosh(u+\tau))\big)\ .$$

Splitting the integral in two parts, making the change of variables $u \to u + \tau$ and $u \to u - \tau$ and using (miracle!) that $\sinh(u+\tau) + \sinh(u-\tau) = 2\sinh u\cosh\tau$ show that $I(x)$ is independent of τ.

Exercise 5.2.2 The first part of the exercise is obvious. To handle the case of the continuous situation, one may like to think in terms of "dimension of operators", how these unit-dependent quantities get rescaled under a change of units. The key point is that the dimension of $a(\boldsymbol{p})$ is $[l^{3/2}]$. To get convinced of this we observe from (3.60) that the square of this dimension is the dimension of $(2\pi\hbar)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$. Now $\int \mathrm{d}^3\boldsymbol{p}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) = 1$, so that $\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$ has the dimension of the inverse of the cube of a momentum, so that $(2\pi\hbar)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$ has dimension $[l^3]$.

Exercise 5.4.1 Let us define the function η on $\mathbb{R}^3$ by $\eta(\boldsymbol{p}) = \xi(\omega_{\boldsymbol{p}}, \boldsymbol{p})/\sqrt{2\omega_{\boldsymbol{p}}}$, so that $\xi = J(\eta)$. Then

$$\int \mathrm{d}\lambda_m(p)\xi(p)^* a(p) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3}\eta(\boldsymbol{p})^* a(\boldsymbol{p}) = A(J(\eta)) = A(\xi) \ .$$

Exercise 5.4.2 (a) makes sense only by integrating against a function $\xi \in \mathcal{S}$. Then, for the right-hand side,

$$\begin{aligned}&\int \mathrm{d}\lambda_m(p)\xi(p)^* \exp(-\mathrm{i}(c, C(p))/\hbar) a(C(p))\\ &= \int \mathrm{d}\lambda_m(p)\xi(C^{-1}(p))^* \exp(-\mathrm{i}(c,p))/\hbar)a(p) = \int \mathrm{d}\lambda_m(p)(U(c,C)(\xi)(p))^* a(p)\\ &= A(U(c,C)(\xi)) = U_{\mathcal{B}}(c,C)\circ A(\xi)\circ U_{\mathcal{B}}(c,C)^{-1} \ .\end{aligned}$$

For (b) we write

$$\begin{aligned}&W(b,B)W(c,C)(a(p)) = \exp(-\mathrm{i}(b, BC(p))/\hbar)\exp(-\mathrm{i}(c, C(p)/\hbar)a(BC(p))\\ &= \exp(-\mathrm{i}(b+B(c), BC(p))/\hbar)a(BC(p)) = W((b,B)(c,C))(a(p)) \ .\end{aligned}$$

(c) To understand this difference meditate on the term $a(C(p))$ (as opposed to $a(C^{-1}(p))$). Here you get a rule to transform the operator $a(p)$ whereas (4.61) is a rule to transform the function φ of p. (d) To deduce Lorentz invariance from (5.42), we write, using (5.41) and the corresponding formula for $a^\dagger$,

$$\begin{aligned}&\sqrt{c}U_{\mathcal{B}}(b,B)\circ\varphi(x)\circ U_{\mathcal{B}}(b,B)^{-1} =\\ &= \int \mathrm{d}\lambda_m(p)\big(\exp(-\mathrm{i}(x,p)/\hbar)\exp(-\mathrm{i}(b,B(p))/\hbar)a(B(p))\\ &\quad + \exp(\mathrm{i}(x,p)/\hbar)\exp(\mathrm{i}(b,B(p))/\hbar)a^\dagger(B(p))\big) \ .\end{aligned}$$

Making the change of variable $p \to B^{-1}(p)$ and using that $(x, B^{-1}(p)) = (B(x), p)$ the right-hand side is $\sqrt{c}\varphi(b + B(x))$.

Exercise 5.4.3 Using (5.39) the formula (5.43) is just another way to write (5.41).

Exercise 5.4.4 Because $\delta_{m,p}$ has the property that $\int \mathrm{d}\lambda_m(p')\delta_{m,p}(p')a(p') = a(p)$.

Exercise 6.1.1 If M is a Lorentz transformation and $x'^\nu = M^\nu{}_\mu x^\mu$ by the chain rule we have

$$\begin{aligned}\eta^{\mu\nu}\partial_\mu\partial_\nu u &= \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}u = \eta^{\mu\nu}\frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial}{\partial x'^\lambda}\frac{\partial x'^\xi}{\partial x^\nu}\frac{\partial}{\partial x'^\xi}u\\ &= \eta^{\mu\nu}M^\lambda{}_\mu M^\xi{}_\nu\frac{\partial}{\partial x'^\lambda}\frac{\partial}{\partial x'^\xi}u = \eta^{\lambda\xi}\frac{\partial}{\partial x'^\lambda}\frac{\partial}{\partial x'^\xi}u \ ,\end{aligned}$$

using that $\eta^{\mu\nu}M^\lambda{}_\mu M^\xi{}_\nu = \eta^{\lambda\xi}$ since M is a Lorentz transformation.

Exercise 6.1.2 Assume for simplicity that u is dimensionless. Then $\partial_\nu u$ is (the limit of) the quotient of an increment of u by a length and is of dimension $[l^{-1}]$. Similarly $\partial^\nu\partial_\nu u$ is of dimension $[l^{-2}]$. Thus the term $\hbar^2\partial^\nu\partial_\nu$ has dimension $[m^2l^4t^{-2}l^{-2}]$ which is the same as the dimension of the term m^2c^2u.

Exercise 6.2.1 No, by replacing a by $\exp(\mathrm{i}t_0\omega)a$.

Exercise 6.4.1 For the motion $x(t) = \cos\omega t$ it is straightforward that the action is zero. But for $x(t) \equiv 1$ the action is < 0. Thus $x(t) = \cos\omega t$ does not minimize the action.

Exercise 6.4.3 We don't know the dimension of u but is irrelevant as the equation is homogeneous in u. The dimension of m^2c^4 is $[m^2l^4t^{-4}]$. The dimension of $\hbar^2c^2$ is $[m^2l^6t^{-4}]$, but each operator ∂_ν creates a dimension $[l^{-1}]$.

Exercise 6.5.3 Let us write $R_{\ell,k}$ the matrix of R, $v_\ell = \sum_{k\le n} R_{\ell,k}\bar{v}_k$. Then, by the chain rule,

$$\bar{p}_k := \frac{\partial\bar{L}}{\partial\bar{v}_k} = \sum_{\ell\le n}\frac{\partial L}{\partial v_\ell}\frac{\partial v_\ell}{\partial\bar{v}_k} = \sum_{\ell\le n} p_\ell\frac{\partial v_\ell}{\partial\bar{v}_k} = \sum_{\ell\le n} p_\ell R_{\ell,k}\ ,$$

and thus $\bar{p} = R^T(p) = R^{-1}(p)$ and $p = R(\bar{p})$. Also,

$$\bar{v} = R^{-1}(v) = R^{-1}(v(x,p)) = R^{-1}(v(R(\bar{x}), R(\bar{p})))\ .$$

If we compute the Hamiltonian in the new coordinates, we obtain

$$\bar{H}(\bar{x},\bar{p}) = \sum_{k\le n}\bar{v}_k\bar{p}_k - \bar{L}(\bar{x},\bar{v})\ .$$

Since $\sum_{k\le n}\bar{v}_k\bar{p}_k = \sum_{k\le n} v_kp_k$ this shows that indeed $\bar{H}(\bar{x},\bar{p}) = H(R(\bar{x}), R(\bar{p}))$.

Exercise 6.5.4 The first statement is just another way to express (6.44). Moreover since the construction of the Hamiltonian $\bar{H}$ is, in the new basis, the same as the construction of H in the old basis, it satisfies Hamilton's equation of motion which is what is meant by the second statement.

Exercise 6.6.1 There is an orthogonal transformation R of $\mathbb{R}^n$ such that the point of old coordinates $x \in \mathbb{R}^n$ has new coordinates $R(x)$. The element f of the state space (when using the new coordinates) is, in the old coordinates, given by $U(f)$ where $U(f)(x) = f(R(x))$. In the new basis the Hamiltonian H_n becomes $U^{-1}H_nU$. It is a simple calculation (using the chain rule) to show that $U^{-1}H_nU = H_n$, the same Hamiltonian as we would have obtained by proceeding to quantization using the new coordinates.

Exercise 6.7.1 Let compute only the hardest term

$$\begin{aligned}\int_B \mathrm{d}^3\boldsymbol{x} \sum_{1\le\nu\le 3} \partial_\nu u(\boldsymbol{x})\partial_\nu u(\boldsymbol{y}) &= -\int_B \mathrm{d}^3\boldsymbol{x} u(\boldsymbol{x}) \sum_{1\le\nu\le 3} \frac{\partial^2}{(\partial x_\nu)^2} v(\boldsymbol{x}) \\ &= -\sum_{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{K}'} b_{\boldsymbol{k}} b_{\boldsymbol{\ell}} \int_B \mathrm{d}^3\boldsymbol{x} g_{\boldsymbol{k}}(\boldsymbol{x}) \sum_{1\le\nu\le 3} \frac{\partial^2}{(\partial x_\nu)^2} g_{\boldsymbol{\ell}}(\boldsymbol{x}) \\ &= \frac{1}{\hbar^2} \sum_{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{K}'} b_{\boldsymbol{k}} b_{\boldsymbol{\ell}} \boldsymbol{\ell}^2 \int_B \mathrm{d}^3\boldsymbol{x} g_{\boldsymbol{k}}(\boldsymbol{x}) g_{\boldsymbol{\ell}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{K}'} \boldsymbol{k}^2 b_{\boldsymbol{k}}^2/\hbar^2 \ ,\end{aligned}$$

using (6.57) in the second line, that $g_{\boldsymbol{\ell}}$ is an eigenvector of the Laplacian, of eigenvalue $-\boldsymbol{\ell}^2/\hbar^2$ in the third line and finally that $(g_{\boldsymbol{k}})$ is an orthogonal basis.

Exercise 6.9.2 In precise terms one is looking for distributions Φ_t such that $\int \mathrm{d}t\Phi_t(f_t) = f(0)$, where $f_t(\boldsymbol{x}) = f(t,\boldsymbol{x})$. Let us then take the function f of the type $f(t,\boldsymbol{x}) = \varphi(t)\psi(\boldsymbol{x})$. Then $\int \mathrm{d}t\varphi(t)\Phi_t(\psi) = \varphi(0)\psi(\boldsymbol{0})$. In particular $\int \mathrm{d}t\varphi(t)\Phi_t(\psi) = 0$ whenever $\varphi(0) = 0$, so that $\Phi_t(\psi) = 0$ whenever $t \neq 0$. Since ψ is arbitrary, $\Phi_t = 0$ when $t \neq 0$. Thus $\int \mathrm{d}t\Phi_t(f_t) = 0$, a contradiction.

Exercise 6.9.3 (a) We compute

$$\partial_t \int \mathrm{d}^3\boldsymbol{x} A \overset{\leftrightarrow}{\partial_t} B = \int \mathrm{d}^3\boldsymbol{x} \partial_t (A \overset{\leftrightarrow}{\partial_t} B) = \int \mathrm{d}^3\boldsymbol{x} (A\partial_t^2 B - B\partial_t^2 A) = 0 \ , \qquad \text{(P.13)}$$

using in the last equality that for $C = A, B$ we have $c^{-2}\partial_t^2 C = -m^2c^2C/\hbar^2 + \sum_{k\le 3} \partial_k^2 C$, and integrating by parts in the space variables. (b) Denoting by B_t the distribution obtained in fixing $x^0 = ct$, and using that A and B satisfy the Klein-Gordon equation the right hand side of (P.13) is $c^{-2}\sum_{k\le 3}\int \mathrm{d}\boldsymbol{x}(A\partial_k^2 B_t - (\partial_k^2 A)B_t)$. Now, by the very definition of the derivative of a distribution, $\int \mathrm{d}\boldsymbol{x} A\partial_k^2 B_t = \int \mathrm{d}\boldsymbol{x}(\partial_k^2 A)B_t$.

Exercise 6.9.4 We observe that

$$2a(\boldsymbol{p}) = \sqrt{2c\omega_{\boldsymbol{p}}}d(\boldsymbol{p}) + \frac{\mathrm{i}}{\hbar}\sqrt{\frac{2}{c\omega_{\boldsymbol{p}}}} f(\boldsymbol{p}) \ ,$$

so that using (6.74) and (6.76), and since $\pi(0,\boldsymbol{x}) = \hbar^2\partial_t\varphi(0,\boldsymbol{x})$, we obtain:

$$\frac{\mathrm{i}}{\hbar}\sqrt{2c\omega_{\boldsymbol{p}}}a(\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x} \exp(-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{p}/\hbar)\Big(\frac{\mathrm{i}}{\hbar}c\omega_{\boldsymbol{p}}\varphi(0,\boldsymbol{x}) - \partial_t\varphi(0,\boldsymbol{x})\Big) \ . \qquad \text{(P.14)}$$

This is the quantity $\int \mathrm{d}^3\boldsymbol{x}\varphi(x) \overset{\leftrightarrow}{\partial_t} \exp(\mathrm{i}(x,p)/\hbar)$ for $x_0 = 0$. To argue that this quantity is independent of x_0 we first have to define it! Thinking of this as a distribution in $\boldsymbol{p}$ we integrate against a test function and we use (b) of the previous exercise.

Exercise 6.9.5 Let us compute e.g. $\int \mathrm{d}^3\boldsymbol{x} A\partial B/\partial t$ when $A =$

$\int \mathrm{d}\lambda_m(p)\exp(\mathrm{i}(x,p)/\hbar)f^+(p)$ and $B = \int \mathrm{d}\lambda_m(p)\exp(\mathrm{i}(x,p)/\hbar)g^+(p)$. Then

$$A\partial B/\partial t = \frac{c\mathrm{i}}{\hbar}\iint \mathrm{d}\lambda_m(p)\mathrm{d}\lambda_m(p')p'^0\exp(\mathrm{i}(x,p+p')/\hbar)f^+(p)g^+(p')\ .$$

Integrating in $\mathrm{d}^3\boldsymbol{x}$ and using the formula $\int \mathrm{d}^3\boldsymbol{x}\exp(\mathrm{i}(\boldsymbol{x},\boldsymbol{p}+\boldsymbol{p}')/\hbar) = (2\pi\hbar)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}')$ yields the formula

$$\int \mathrm{d}^3\boldsymbol{x}A\partial B/\partial t = \frac{c\mathrm{i}}{\hbar}\int \mathrm{d}\lambda_m(p)p^0\exp(\mathrm{i}x_0p^0/\hbar)f^+(p)g^+(p') = \int \mathrm{d}^3\boldsymbol{x}\partial A/\partial tB\ .$$

Proceeding in a similar fashion for the other terms one gets

$$\int \mathrm{d}^3\boldsymbol{x}A \overset{\leftrightarrow}{\partial_t} B = \frac{2c\mathrm{i}}{\hbar}\int \mathrm{d}\lambda_m(p)p^0(f^-(p)g^+(p) - g^-(p)f^+(p))\ .$$

Exercise 6.11.2 First one has $\mathcal{H}(\varphi,\hbar^2\partial_t\varphi) = \frac{1}{2}\hbar^2c^2\sum_{0\le\nu\le 3}(\partial_\nu\varphi)^2 + \frac{1}{2}m^2c^4\varphi^2$.

We compute the integral of each of the first four terms as in (6.82). The tricky part is that implementing the condition $\delta(\boldsymbol{r}) = 0$ for $\boldsymbol{r} = \pm\boldsymbol{p}\pm\boldsymbol{p}'$ does not affect the term $p_\nu p'_\nu$ the same way when $\nu = 0$ and when $\nu \ge 1$ because $\omega_{-\boldsymbol{p}} = \omega_{\boldsymbol{p}}$. When $\nu = 0$ one gets

$$\hbar^2\int \mathrm{d}^3\boldsymbol{x}(\partial_0\varphi(x))^2 = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3 2c\omega_{\boldsymbol{p}}}(p_0)^2\Big(-\exp(-2\mathrm{i}\omega_{\boldsymbol{p}}ct/\hbar)a(\boldsymbol{p})a(-\boldsymbol{p}) + a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) - \exp(2\mathrm{i}\omega_{\boldsymbol{p}}ct/\hbar)a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\Big)\ ,$$

whereas for $\nu \ge 1$

$$\hbar^2\int \mathrm{d}^3\boldsymbol{x}(\partial_\nu\varphi(x))^2 = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3 2c\omega_{\boldsymbol{p}}}(p_\nu)^2\Big(\exp(-2\mathrm{i}\omega_{\boldsymbol{p}}ct/\hbar)a(\boldsymbol{p})a(-\boldsymbol{p}) + a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + \exp(2\mathrm{i}\omega_{\boldsymbol{p}}ct/\hbar)a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\Big)\ .$$

Using the relation $m^2c^2 = p^\mu p_\mu$, and since $p^0 = \omega_{\boldsymbol{p}}$, algebra then yields

$$\int \mathrm{d}^3\boldsymbol{x}\mathcal{H}(\varphi(x),\hbar^2\partial_t\varphi(x)) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3 2}c\omega_{\boldsymbol{p}}\big(a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\big)\ ,$$

which is just (5.36) modulo the infinite term obtained when replacing $a(\boldsymbol{p})a^\dagger(\boldsymbol{p})$ by $a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + \delta^{(3)}(\boldsymbol{0})1$.

Exercise 8.1.10 Consider a positive Hermitian matrix B with $B^2 = A$. Consider an orthonormal basis in which B is diagonal. When $A = B^2$ is a multiple of the identity then so is B which is then clearly unique. Otherwise, the eigenvectors of A are uniquely determined and again B is uniquely determined. That makes the continuity of the map $A \mapsto B$ obvious by a standard subsequence argument.

Exercise 8.1.5 Taking the complex conjugate of the relation $J^{-1}C^*J = C^{\dagger -1}$ yields $J^{-1}CJ = C^{T-1}$.

Exercise 8.2.2 Just integrate over $\mathbb{S}$ the relation $\sum_{0\le i\le j}\binom{j}{i}|z_1^i z_2^{j-i}|^2 = 1$.

Exercise 8.2.4 Let us identify the space $\mathcal{H}_1$ of homogeneous first degree polynomials to the set $\mathbb{C}^2$ seen as a set of column matrices. That is to the matrix $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ corresponds the polynomial $f_a(z_1, z_2) = a_1 z_1 + a_2 z_2 = a^T z$ where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Thus $\pi_1(A)(f_a)(z_1, z_2) = f_a(z_1', z_2') = a^T A^\dagger(z) = (A^* a)^T z$. This means that $\pi_1(A)(f_a) = f_{A^*a}$. In other words π_1 is equivalent to the representation of π of $SU(2)$ on $\mathbb{C}^2$ where $\pi(A)$ is the operator with matrix A^*. According to Lemma 8.5.4 this representation is equivalent to the representation π' of $SU(2)$ where $\pi'(A)$ is the operator with matrix $A^{\dagger -1} = A$.

Exercise 8.2.8 It is obvious that $\alpha_i = \alpha_{j-i}$, and the proof of (8.12) is then straightforward from the definitions. For example, in the case of J we have $z_1' = -z_2$ and $z_2' = z_1$. Within a multiplicative factor which is a positive number and is due to the normalization constants α_i, $\pi_j(C)_{k,\ell}$ is the coefficient of $z_1^k z_2^{j-k}$ in the expansion of $z_1'^{\ell} z_2'^{j-\ell}$ where z_1' and z_2' are given by (8.7), and this should make the first equality in (8.13) obvious. Next, we use that $C^* = J^{-1}CJ$ so that $\pi_j(C^*) = \pi_j(J^{-1})\pi_j(C)\pi_j(J)$. We then compute $\pi_j(C^*)(f_i)$ using (8.12) and the formula $\pi_j(C)(f_k) = \sum_\ell \pi_j(C)_{\ell,k} f_\ell$ to obtain the result.

Exercise 8.4.1 We use the definition of κ. The point $y = \kappa(\exp(a\sigma_3))(x)$ is such that $M(y) = \exp(a\sigma_3)M(x)\exp(a^*\sigma_3)$, using also that $\exp(a\sigma_3)^\dagger = \exp(a^*\sigma_3)$. This gives the relations

$$y^0 + y^3 = \exp(a + a^*)(x^0 + x^3)\ ;\ y^0 - y^3 = \exp(-a - a^*)(x^0 + x^3)$$

$$y^1 + \mathrm{i}y^2 = \exp(-a + a^*)(x^1 + \mathrm{i}x^2)\ ;\ y^1 - \mathrm{i}y^2 = \exp(a - a^*)(x^1 - \mathrm{i}x^2)\ .$$

Wheb $a = s/2 \in \mathbb{R}$ this coincides with the formula (4.21) for B_s. When $a = -\mathrm{i}\theta/2$ for $\theta \in \mathbb{R}$ this gives the relations $y^0 = x^0, y^3 = x^3$, $y^1 + \mathrm{i}y^2 = \exp(\mathrm{i}\theta)(x^1 + \mathrm{i}x^2)$ and $y^1 - \mathrm{i}y^2 = \exp(-\mathrm{i}\theta)(x^1 - \mathrm{i}x^2)$ i.e. $y^1 = x^1\cos\theta - x^2\sin\theta$ and $y^2 = x^1\sin\theta + x^2\cos\theta$ which indeed correspond to a rotation of angle θ in the plane spanned by e_1 and e_2.

Exercise 8.4.2 Since $\kappa(SL(2,\mathbb{C}))$ contains all pure boosts it suffices from (8.22) to prove that it contains all rotations. From (8.23) $\kappa(\exp(-\mathrm{i}\theta\sigma_3/2)$ is a rotation of angle θ around the third axis. Since κ is 2-to-1, if an element $A \in SU(2)$ is not of the type $\exp(-\mathrm{i}\theta\sigma_3/2)$, $\kappa(A)$ is a rotation around another axis than the third axis. Thus $G := \kappa(SU(2))$ is a group of rotations which contains all rotations around the third axis, and at least another rotation. We will show that such a group must be the entire group rotations. The set $\mathcal{D}$ of directions with the property that any rotation around this direction belongs to G is invariant under the action of any rotation of G. This is because if R and S are rotations, RSR^{-1} is a rotation of the same angle as S but around the image by R of the axis of rotation of S. Thus $\mathcal{D}$ contains the direction of the third axis, and another direction D_0, which after rotation around the third axis, we may assume to be in the plane generated by e_1 and e_3. Let us call θ the angle between D_0 and e_3. By rotating D_0 around the third axis, to a direction D with an angle ξ with D_0, $0 \le \xi \le 2\theta$, and then bringing D back to the e_1, e_3 plane by a rotation around D_0 we can obtain every direction in the e_1, e_3 plane with an angle less than 3θ with e_3, and the conclusion should then be obvious.

Exercise 8.4.4 Look at the solution of Exercise D.12.4

Exercise 8.4.5 Unfortunately a direct proof of that does not seem much easier than the solution of Exercise 8.4.4. It is much easier to appeal to Theorem D.6.4. The only thing we have show is that the representation is irreducible. When there is an invariant subspace in a unitary representation, the orthogonal complement of the subspace is also invariant. Since here the representation lives in a space of dimension 3, if it was not irreducible there would be a one-dimensional invariant subspace, but it is quite obvious that such a subspace does not exist.

Exercise 8.4.7 Let us first observe that a pure boost L satisfies $L = L^T$. This is obvious if $L = B^s$, and in general $L = RB^sR^{-1}$ for a rotation R so that $L^T = (R^{-1})^T B^s R^T = L$. Thus if A is positive Hermitian, $\kappa(A^\dagger) = \kappa(A) = \kappa(A)^T$ because $\kappa(A)$ is a pure boost. On the other hand, if A is unitary,

$$\kappa(A^\dagger) = \kappa(A^{-1}) = \kappa(A)^{-1} = \kappa(A)^T ,$$

because $\kappa(A)$ is a rotation. Thus it suffices to prove that every A in $SL(2,\mathbb{C})$ is of the type $A = VU$ where V is positive Hermitian and U is unitary[2] and since then $\kappa((VU)^\dagger) = \kappa(U^\dagger)\kappa(V^\dagger) = \kappa(U)^T\kappa(V)^T = \kappa(VU)^T$. It is plain that $AA^\dagger$ is positive Hermitian, so that there is an orthonormal basis in which it is diagonal. In that basis it is obvious that it can be written as $VV^\dagger$ where V is positive Hermitian: the diagonal entries of V are simply the positive square roots of the eigenvalues of $AA^\dagger$. It then suffices to show that $U := V^{-1}A$ is unitary because $A = VU$. But $UU^\dagger = V^{-1}AA^\dagger V^{\dagger -1}$ is the identity since $AA^\dagger = VV^\dagger$.

Exercise 8.5.3 The construction is described e.g. in [30] page 499 or in [84] page 346.

Exercise 8.5.4 We recall that $\rho(AB) = \pm\rho(A)\rho(B)$. Obviously when $\rho(AB) = \rho(A)\rho(B)$ then $\pi'(AB) = \pi'(A)\pi'(B)$. On the other hand when

$$\rho(AB) = -\rho(A)\rho(B) = (-I)\rho(A)\rho(B)$$

then $\pi'(AB) = \pi(-I)\pi'(A)\pi'(B)$.

Exercise 8.5.5 The group $SO(3)$ naturally acts on $\mathbb{C}^3$ by multiplication of a matrix

[2] This is often called the polar decomposition of A.

by a column vector. This defines a unitary representation of $SO(3)$ in dimension three, and it is easy to prove that it is irreducible. What else could we obtain by using Lemma 8.5.4 for π_2? Proving that this is actually the case is a different matter. We present a self-contained argument, which of course is difficult to invent if one does not know some general theory. Consider the space $\mathcal{M}$ of 2×2 complex matrices with trace (sum of the diagonal coefficients) zero and the representation θ of $SU(2)$ such that $\theta(A)$ is the operator $M \to AMA^\dagger$. Let us show that this representation is equivalent to π_2. Recalling the matrix J of Lemma 8.1.4, let us denote by $\mathcal{M}'$ the spaces of matrices of the type MJ for $M \in \mathcal{M}$. Using the map $M \to MJ^{-1}$ from $\mathcal{M}'$ to $\mathcal{M}$ shows that θ is equivalent to the representation θ of $SU(2)$ on $\mathcal{M}$ such that $\theta'(A)$ is the map $M \to AMJ^{-1}A^{-1}J$ from $\mathcal{M}'$ to itself. In Lemma 8.1.4 we prove that for $C \in SU(2)$ we have $J^{-1}C^*J = C^{\dagger -1} = C$, so that $J^{-1}A^\dagger J = A^T$, and $\theta'(A)$ is the map $M \to AMA^T$. Now it is straightforward to see that $\mathcal{M}'$ consists of symmetric matrices so it identifies with the space of symmetric order two tensors, and under this identification writing the formula for $\theta'(A)$ shows that θ' is just π_2. Thus θ is equivalent to π_2. We now have to prove that the representation $B \to \theta(\rho(B))$ is equivalent to matrix multiplication by B.

Consider the map $U : \mathbb{C}^3 \to \mathcal{M}$ given by

$$U(x^1, x^2, x^3) = \begin{pmatrix} x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & -x^3 \end{pmatrix} .$$

By definition of the map κ, for $A \in SU(2)$, $U^{-1}\theta(A)U$ is the restriction of $\kappa(A)$ to $\mathbb{R}^3$. When D is a rotation of $\mathbb{R}^4$ then $\kappa(\rho(D))$ is simply D, so that when B is a rotation of $\mathbb{R}^3$, seen as a rotation of $\mathbb{R}^3$, B is the restriction of $U^{-1}\theta(\rho(B))U$ to $\mathbb{R}^3$.

Exercise 8.5.6 Since $-I$ commutes with every operator A, $\pi(-I)$ commutes with every $\pi(A)$ so that from Schur's lemma it is a multiple $c1$ of the identity. Since $\pi(-I)^2 = 1$ we have $c^2 = 1$. The rest is obvious from Lemma 8.5.4.

Exercise 8.5.7 If $A \in \kappa^{-1}(B)$ and $A' \in \kappa^{-1}(C)$ then $\kappa(AA') = \kappa(A)\kappa(A') \in BC$.

Exercise 8.6.3 Using (8.32) when $A = \rho(C)$ where ρ is as in Section 8.5 and recalling (8.25) we observe in particular that

$$\forall C \in SO(3) \; ; \; \pi'(C) = \lambda'(C)\pi(\rho(C)) \tag{P.15}$$

where $\lambda'(C) := \lambda(\rho(C))$ is of modulus 1. In words, all projective representations of $SO(3)$ arise from a projective representation of $SU(2)$ obtained through the procedure of Exercise 8.5.4.

Exercise 8.8.1 The computation is done on page 713.

Exercise 8.8.2 The wave function is an element $\varphi \in L^2 \otimes \mathcal{H}_j$ i.e. a function $\varphi : \mathbb{R}^3 \to \mathcal{H}_j$. The hypothesis that measurement of the spin always yields the value $\hbar(j/2 - k)$ means that this wave function is an eigenvector of the spin operator $1 \otimes S_3$ with eigenvalue $\hbar(j/2 - k)$, so that for each $\boldsymbol{p} \in \mathbb{R}^3$, $\varphi(\boldsymbol{p})$ is an eigenvector of the operator S_3 with eigenvalue $\hbar(j/2-k)$. According to the definition (8.35) of S_3 this implies that $\pi_j(\exp(-\mathrm{i}\theta\sigma_3/2))(\varphi(\boldsymbol{p})) = \exp(\mathrm{i}(j/2-k)\theta)\varphi(\boldsymbol{p})$. Next, the action of an element A of $SU(2)$ on the wave function is given by (8.34), and the element A corresponding to a rotation of angle θ around the z axis is $A = \exp(-\mathrm{i}\theta\sigma_3/2))$. In the ideal case where $\varphi(\boldsymbol{p})$ is not zero only when $\boldsymbol{p}$ is in the direction of the z axis, $\varphi(\kappa(A)(\boldsymbol{p})) = \varphi(\boldsymbol{p})$ and then $\pi_j(A)(\varphi(\kappa(A)(\boldsymbol{p}))) = \pi_j(A)(\varphi(\boldsymbol{p}))) = \exp(\mathrm{i}(j/2 - k)\theta)\varphi(\boldsymbol{p})$ which is the desired result.

Exercise 8.9.4 Denoting by $S(A)$ the first matrix and by S' the second one, this should be obvious from the relations $S(A)S(B) = S(AB)$ and $S'S(A)S' = S(A^{\dagger -1})$.

Exercise 8.10.1 (a) Just reverse the manipulations leading from (8.47) to (8.51). (b) Taking determinants in the relation $\gamma(\theta(L)(x)) = L\gamma(x)L^{-1}$ proves that $\theta(L) \in O(1,3)$. To prove that $Pin(1,3)$ is a group, consider two elements L, L' of this set. Then $L'L\gamma(x)L^{-1}L'^{-1} = L'\gamma(\theta(L)(x))L'^{-1} = \gamma(\theta(L')\theta(L)(x))$ so that $L'L \in Pin(1,3)$ and $\theta(L'L) = \theta(L')\theta(L)$ (etc.) (c) It is better to take for granted that the linear span of the product of γ matrices is the set of all matrices, as there is nothing to learn from checking that. Prove then that a matrix which commutes with every matrix is a multiple of the identity, which is straightforward. (d) It follows from (8.47) that $S(A) \in Pin(1,3)$ and that $\theta(S(A)) = \kappa(A)$. Thus $\theta(S(SL^+(2,\mathbb{C}))) = \kappa(SL^+(2,\mathbb{C}))$. And we proved in Section 8.9 that $\kappa(SL^+(2,\mathbb{C})) = O^+(1,3)$. (e) It is obvious that S is one-to-one. We have already proved that it is an homomorphism from $SL^+(2,\mathbb{C})$ into $\theta^{-1}(O^+(1,3))$, and we have to prove that it is onto. According to (c) given any $C \in O^+(1,3)$ we can find $D \in SL^+(2,\mathbb{C})$ with $\theta(S(D)) = C$. Let us define $-D$ in the obvious manner if $D \in SL(2,\mathbb{C})$ and $-D = P'(-A)$ if $D = P'A$ for $A \in SL(2,\mathbb{C})$. Then $\kappa(-D) = \kappa(D)$, so that $\theta(S(D)) = \theta(S(-D))$. This means that we have found two different elements of $S(SL^+(2,\mathbb{C}))$ whose image by θ is C. Then (d) implies that $S(SL^+(2,\mathbb{C}))$ contains $\theta^{-1}(A)$. (f): It is straightforward to compute that $\theta(T)^0{}_0 = -1, \theta(T)^i{}_i = 1$ for $1 \le i \le 3$, the other being zero. This "reverses the flow of time". (g) We may guess the rules from the previous matrix expressions: $T'^2 = -1$, $P'T' = T'P'$, $T'A = -A^{\dagger -1}T'$. Let us denote by $SL(2,\mathbb{C})^*$ the group generated by $SL^+(2,\mathbb{C})$ and T', and extend S to $SL(2,\mathbb{C})^*$ by defining $S(T') = T$. We then prove as in (e) that S is an isomorphism from $SL(2,\mathbb{C})^*$ to $Pin(1,3)$.

Exercise 8.10.2 (a) The equivalence is given by the map $S(P')$ since $S(A^{\dagger -1}) = S(P')S(A)S(P')^{-1}$. (b) It should be transparent that if S is the representation (j,ℓ) of $SL(2,\mathbb{C})$ then $A \mapsto S(A^{\dagger -1})$ is the representation (ℓ, j) and these can be the equivalent only if $j = \ell$. $S(P')$ then has to exchange the actions of A and $A^\dagger$. I see two maps which achieve this, namely the map $(x_{i_1,\ldots,i_\ell,j_1,\ldots,j_\ell}) \mapsto (x_{j_1,\ldots,j_\ell,i_1,\ldots,i_\ell})$ and the map $(x_{i_1,\ldots,i_\ell,j_1,\ldots,j_\ell}) \mapsto (-x_{j_1,\ldots,j_\ell,i_1,\ldots,i_\ell})$ but I could not prove that there are no others.

Exercise 8.10.4 (a) This is because $S(A)S(P) = S(P)S(A^{\dagger -1})$. (b) Indeed the space $\mathcal{G}\cap\mathcal{G}'$ is invariant under S so that it reduces to $\{0\}$. (c) Since the space $\mathcal{G}\oplus\mathcal{G}'$ is invariant under S and since S is irreducible. (d) Let us denote by θ the restriction of S to $SL(2,\mathbb{C})$ and $\mathcal{G}$. It is irreducible because if $\mathcal{K}$ is an invariant subspace of $\mathcal{G}$ for θ then $\mathcal{K}\oplus S(P')(\mathcal{K})$ is an invariant subspace of S. Let us then consider the map $T:\mathcal{G}\oplus\mathcal{G}\to\mathcal{H}=\mathcal{G}\oplus\mathcal{G}'$ given by $T(x,y)=x+S(P')(y)$. Using again $S(A)S(P')=S(P')S(A^{\dagger-1})$ one obtains the relations $T^{-1}S(A)T(x,y)=(\theta(A)(x),\theta(A^{\dagger-1})(x))$ and $T^{-1}S(P')T(x,y)=(y,x)$.

Exercise 9.1.1 The point is that then $\varphi(A^{-1}(p))\neq 0$ only for $p=A(p')$.

Exercise 9.2.1 We have $(-A)(b)=\kappa(-A)(b)=\kappa(A)(b)=A(b)$.

Exercise 9.2.3 The point is that the element $(0,-I)$ commutes with every element (a,A) because $-Ia=\kappa(-I)a=a$. Thus by Shur's lemma $\pi(0,-I)$ is a multiple of the identity. In fact, it is $\pm$ the identity since its square is the identity. And when $\rho(A)\rho(B)=-\rho(AB)$ we have $(a,\rho(A))(b,\rho(B))=(a+\rho(A)(b),\rho(A)\rho(B))=(a+Aa,\rho(AB))(0,-I)$ since $\rho(A)a=\kappa(\rho(A))(a)$.

Exercise 9.4.3 Since $(D_p)^{-1}D'_p)$ belongs to the little group, on which V is unitary, the operator $V(D_p^{-1}D'_p)$ is unitary. Now, using (9.7) we obtain

$$U(a,A)(W(\varphi))(p)=\exp(\mathrm{i}(a,p)/\hbar)V(D_p^{-1}AD_{A^{-1}(p)})[W(\varphi)(A^{-1}(p))]\ ,$$

and since $W(\varphi)(A^{-1}(p))=V(D^{-1}_{A^{-1}(p)}D'_{A^{-1}(p)})\varphi(A^{-1}(p))$ the right-hand side is, with obvious notation,

$$V(D_p^{-1}D'_p)\Big(\exp(\mathrm{i}(a,p)/\hbar)V(D'^{-1}_p AD'_{A^{-1}(p)})[\varphi(A^{-1}(p))]\Big)=W(U'(a,A)(\varphi))(p)\ .$$

Exercise 9.5.5 (a) We have $\int \mathrm{d}\mu(A)f(CA)=\iint \mathrm{d}\lambda_m(p)\mathrm{d}\nu(B)f(CD_pB)$. Now, $CD_p=D_{C(p)}D$ where $D\in SU(2)$, so that by left-invariance of $\mathrm{d}\nu$ we have $\int \mathrm{d}\nu(B)f(CD_pB)=\int \mathrm{d}\nu(B)f(D_{C(p)}B)$ and thus $\iint \mathrm{d}\lambda_m(p)\mathrm{d}\nu(B)f(CD_pB)=\iint \mathrm{d}\lambda_m(p)\mathrm{d}\nu(B)f(D_{C(p)}B)=\int \mathrm{d}\mu(A)f(A)$ as is shown by the change of variables $p\to C^{-1}(p)$ and the invariance of $\mathrm{d}\lambda_m$. (b) Since $D_pB(p^*)=D_pp^*=p$ we have

$$\int \mathrm{d}\mu(A)\|V(A)^{-1}\varphi(A(p^*))\|^2=\iint \mathrm{d}\lambda_m(p)\mathrm{d}\nu(B)\|V(B)^{-1}V(D_p)^{-1}(\varphi(p))\|^2\ .$$

Now $V(D_p)^{-1}(\varphi(p))\in\mathcal{V}$ and since $V(B)^{-1}$ is unitary on $\mathcal{V}$ we have $\|V(B)^{-1}V(D_p)^{-1}(\varphi(p))\|^2=\|V(D_p)^{-1}(\varphi(p))\|^2=\|\varphi(p)\|_p^2$. Thus $\int \mathrm{d}\mu(A)\|V(A)^{-1}\varphi(A(p^*))\|^2=\iint \mathrm{d}\lambda(p)\mathrm{d}\nu(B)\|\varphi(p)\|_p^2=\int \mathrm{d}\lambda_m(p)\|\varphi(p)\|_p^2=\|\varphi\|^2$. (c) There still exists a left-invariant measure on the little group, but it cannot be a probability because the little group is not compact, and the previous argument does not work.

Exercise 9.5.8 By definition $\|u\|_{R,p}=\|D_p^{-1}u\|$ and $\|u\|_{L,p}=\|D_p^{\dagger}u\|$. We assume that $D_p=D_p^{\dagger}$ so that $D_p^{\dagger}D_p^{-2}=D_p^{-1}$. Thus $\|W(\varphi)(p)\|_{L,p}=\|\varphi(p)\|_{R,p}$. Now,

$$U_L(a,A)W(\varphi)(p)=\exp(\mathrm{i}(a,p)/\hbar)A^{\dagger-1}D^{-2}_{A^{-1}(p)}\varphi(A^{-1}(p))$$

$$WU_R(a,A)(\varphi)(p) = \exp(\mathrm{i}(a,p)/\hbar)D_p^{-2}A\varphi(A^{-1}(p))$$

and these quantities are equal since $A^{\dagger -1}D_{A^{-1}(p)}^{-2} = D_p^{-2}A$ as a consequence of the fact that $B := D_p^{-1}AD_{A^{-1}(p)} \in SU(2)$ so that $B = B^{\dagger -1}$.

Exercise 9.6.5 One of the goals of this exercise is to identify the little group as a semi-direct product. This allows one to find its representations. (a) There is no difficulty checking that this is a group, e.g. $(c,a)^{-1} = (-ca^{-2}, a^{-1})$, and (b) is also straightforward. The "double cover" arises from the fact that $(c, \pm a)$ correspond to the same transformation. Denoting by $A(a,b)$ the matrix (9.40) it holds $A(a,b)A(a',b') = A(aa', ab' + ba'^*)$ and indeed

$$(aa'(ab' + ba'^*), aa') = (a^2a'b' + ab, aa') = (ab, a)(a'b', a') \ .$$

For (d) observe that unitarity is obvious as $|\exp(\mathrm{i}\,\mathrm{Im}\,(\alpha b^* w))| = 1$ and simply write

$$U(c',a')U(c,a)(f)(w) = (aa')^j \exp(\mathrm{i}\,\mathrm{Im}\,(\alpha c'^* w))\exp(\mathrm{i}\,\mathrm{Im}\,(\alpha c^* a'^{-2}w))f((aa')^{-2}w) \ ,$$

and this is $U((c',a')(c,a))(f)(W)$ since $c'^* + c^*a'^{-2} = (c' + a'^2 c)^*$.

Exercise 9.6.8 (a) and (b) are straightforward. So is (c). Indeed, for $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,\mathbb{C})$ we have $B^{-1} = \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} \in SL(2,\mathbb{C})$, $B^{-1}\cdot z = (-c' + a'z)/(d' - b'z)$, $f(B,z) = d' - b'z$, $f(A, B^{-1}\cdot z) = d - b(-c' + a'z)/(d' - b'z)$ and $f(BA,z) = dd' + bc' - (db' + ba')z = f(B,z)f(A, B^{-1}\cdot z)$. (d) For $p \in X_0 \setminus \{0\}$, we have $M(p) = (p^0 + p^3)ZZ^\dagger$ where Z is the column matrix $\begin{pmatrix} 1 \\ z \end{pmatrix}$ for $z = (p^1 + \mathrm{i}p^2)/(p^0 + p^3)$. The map $p \mapsto z = (p^1 + \mathrm{i}p^2)/(p^0 + p^3)$ provides an identification of the quotient of $X_0 \setminus \{0\}$ by the equivalence relation $p\mathcal{R}p'$ if $p' = \lambda p$ for some $\lambda > 0$ identifies with $\mathbb{C}$ and the action of $SL(2,\mathbb{C})$ on $X_0 \setminus \{0\}$ respects this equivalence relation. The quotient action of $SL(2,\mathbb{C})$ on $\mathbb{C}$ is the one we study here, and (9.46) implies (9.45) for the function occurring in (9.42). .

Exercise 9.6.9 (a) is obvious. It is obvious that $\mathcal{H}_j$ is invariant under the transformations $V(a,A)$. To prove (b), writing what this means boils down to proving that $p(Av) = A(p(v))$. This is because $M(p(Av)) = Avv^\dagger A^\dagger = AM(p(v))A^\dagger = M(A(p(v)))$. (c) It is obvious that $w(p)w(p)^\dagger = M(p)$ so that $p(w(p)) = p$. When $vv^\dagger = M(p)$ then $v = \theta w(p)$ where $|\theta| = 1$. Then $\theta = v_1/|v_1|$ since $w(p)_1 \geq 0$. (d) we have $(Bw(p))(Bw(p))^\dagger = Bw(p)w(p)^\dagger B^\dagger = BM(p)B^\dagger = M(B(p))$ and the result by (c). (e) We compute

$$TU(a,A)(f)(p) = \exp(\mathrm{i}(a, p(w(p)))f(A^{-1}w(p)) \ . \tag{P.16}$$

Now, by (d) we have $A^{-1}w(p) = \theta w(A^{-1}(p))$ for $\theta = A^{-1}(w(p))_1/|A^{-1}(w(p))_1|$, so that $f(A^{-1}w(p)) = \theta^j f(w(A^{-1}(p)))$. Writing what this means makes it obvious that $\theta = \xi(A, w(p))/|\xi(A, w(p)|$, so that the right-hand side of (P.16) is $V(a,A)T(f)$. Finally we show that T is proportional to a unitary map. For this we note that

Lebesgue's measure on $\mathbb{C}^2$ is invariant by the action of $SL(2,\mathbb{C})$. Since $p(Av) = Ap(v)$, the image of Lebesgue's measure by the map $v \mapsto p(v)$ is a measure on X_0 which is invariant under the action of $SL(2,\mathbb{C})$. Thus this image is proportional to λ_0.

Exercise 9.6.11 One replaces $V(A)$ by $V(A^{\dagger-1})$ and g by the tensor g' such that $g'_{n_1,\ldots,n_j} = 0$ unless all indices equal two, in which case $g_{n_1,\ldots,n_j} = 1$. Formula (9.48) is unchanged. (It would also be possible to keep the same tensor g and to replace $V(A)$ buy $V(A^*)$, but the formulas in the next exercise are not as clean.)

Exercise 9.6.14 One shows that (9.50) has to be replaced by $\mathcal{V}_p = \{u \in \mathcal{H}_0 = \mathbb{C}^2, M(p)(u) = 0\}$. One replaces (9.52) by

$$U'(a,A)(\varphi)(p) = \exp(\mathrm{i}(a,p)/\hbar)A^{\dagger-1}[\varphi(A^{-1}(p))] \tag{P.17}$$

on the space $\mathcal{H}'$ of functions φ which satisfy $M(p)(\varphi(p)) = 0$, provided with the norm (9.48). see previous exercise.

Exercise 9.6.15 In the case $p = p^*$ the relation $(M(Pp^*)\otimes I\otimes\cdots\otimes I)(u) = 0$ means that all components $u_{i_1,\ldots,i_j}$ of u are zero unless $i_1 = 1$ and since u is a symmetric tensor the only non-zero component is for $i_1 = \ldots = i_j = 1$. Thus $u \in \mathcal{V}$ if and only if $(M(Pp^*)\otimes I\otimes\cdots\otimes I)(u) = 0$. For the general case an element u belongs to $\mathcal{V}_p$ if and only if $u = D_p^{\otimes j}(v)$ for $v \in \mathcal{V}_{p^*}$. Consider the operator $U = D_p M(Pp^*)D_p^{-1}$. Then U satisfies $UD_p = D_p M(Pp^*)$ so that $(U\otimes I\otimes\cdots\otimes I)(u) = D_p^{\otimes j}(M(Pp^*)\otimes\cdots\otimes I)(v)$, and hence $u \in \mathcal{V}_p$ if and only if $(U\otimes I\otimes\cdots\otimes I)(u) = 0$. Finally $U = D_p D_p^\dagger M(Pp)$ since $M(Pp) = D_p^{\dagger-1}M(Pp^*)D_p^{-1}$ by (9.51).

Exercise 9.7.2 I find striking that in the massless case all these representations live on the same Hilbert space, see Proposition 9.6.6.[3]

Exercise 9.7.3 The point of (a) is simply that $\kappa(\exp(-\mathrm{i}\theta\boldsymbol{u}\cdot\boldsymbol{\sigma}/2)) = R(\theta,\boldsymbol{u})$. For (b), recalling that Ap stands for $\kappa(A)(p)$, when $\boldsymbol{u} = \boldsymbol{p}/|\boldsymbol{p}|$ and $A = \exp(-\mathrm{i}\theta\boldsymbol{u}\cdot\boldsymbol{\sigma}/2)$ we have $A^{-1}(p) = p$ so that from (9.52) and since $W(A) = U(0,A)$

$$W(\kappa(\exp(-\mathrm{i}\theta\boldsymbol{u}\cdot\boldsymbol{\sigma}/2))(\varphi)(p) = \exp(-\mathrm{i}\theta\boldsymbol{u}\cdot\boldsymbol{\sigma}/2)\varphi(p)$$

and from (9.60) we get (9.61). For (c) since $M(p) = p^\mu\sigma_\mu$ we have $M(Pp) = p^0\sigma_0 - \boldsymbol{p}\cdot\boldsymbol{\sigma}$. Since $p \in X_0$ we have $p^0 = |\boldsymbol{p}|$. Moreover, since $\boldsymbol{u} = \boldsymbol{p}/|\boldsymbol{p}|$ the relation $M(Pp)(\varphi)(p) = 0$ implies $\boldsymbol{u}\cdot\boldsymbol{\sigma}(\varphi(p)) = \varphi(p)$. Hence (9.61) implies $J_{\boldsymbol{u}}(\varphi)(p) = -\hbar/2\varphi(p)$. That is, φ is an eigenvector of the momentum operator $J_{\boldsymbol{u}}$. It represents a state which has a momentum (=$\hbar$ times spin) $-\hbar/2$ along the direction of $\boldsymbol{u}$.

Exercise 9.8.2 To check the algebra, we set

$$g(C) := \pi(a,A)(f)(C) = \exp(\mathrm{i}(C^{-1}(a),p^*)/\hbar)f(A^{-1}C)\ , \tag{P.18}$$

[3] Well, all the Hilbert spaces are the same, but I am sure you see what I mean.

so that

$$\begin{aligned}\pi(b,B)\pi(a,A)(f)(C) &= \pi(b,B)(g)(C) = \exp(\mathrm{i}(C^{-1}(b),p^*)/\hbar)g(B^{-1}C)\\ &= \exp(\mathrm{i}(C^{-1}(b)+C^{-1}(B(a)),p^*)/\hbar)f(A^{-1}B^{-1}C)\end{aligned}$$

and this is indeed $\pi((b,B)(a,A))(f)(C)$. Consider $C,D \in SL(2,\mathbb{C})$ with $C(p^*) = D(p^*)$. Taking $A = D, B = D^{-1}C$ in (9.66) yields

$$f(C) = V(D^{-1}C)^{-1}f(D) = V(C^{-1}D)f(D)\ . \tag{P.19}$$

In particular $\|f(C)\| = \|f(D)\|$, so that $\|f(D_p)\|$ is independent of the choice of D_p. This shows also that $\|f(A^{-1}D_p)\| = \|f(D_{A^{-1}(p)})\|$, and from this it is straightforward to check that $\pi(a,A)$ is unitary.

Exercise 9.8.3 Proceeding as in Exercise 9.5.5 (b) we write $\int \mathrm{d}\mu(A)\|f(A)\|^2 = \int \mathrm{d}\lambda_m(p)\mathrm{d}\nu(B)\|f(D_pB)\|^2$. Now by (9.66) we have $f(D_pB) = V(B^{-1})f(D_p)$ and since $V(B^{-1})$ is unitary we have $\|f(D_pB)\| = \|f(D_p)\|$ and the result follows.

Exercise 9.8.4 If instead of p^* we use the specific point $C(p^*)$ and a representation V' of the little group G' of $C(p^*)$ the state space is the space $\mathcal{F}'$ of functions $f : SL(2,\mathbb{C}) \to \mathcal{V}$ for which $f(AB) = V'(B)^{-1}f(A)$ for $A \in SL(2,\mathbb{C})$ and $B \in G'$, and the representation is given by $\pi(a,A)(f)(B) = \exp(\mathrm{i}(a, BC(p^*))f(A^{-1}B)$. When $V'(B) = V(C^{-1}BC)$ for $B \in G'$, an intertwining map T from the space $\mathcal{F}$ of functions which satisfies the condition (9.66) to $\mathcal{F}'$ is given by $T(f)(A) = f(AC)$. The details are straightforward.

Exercise 9.8.5 Please read Section A.4. One can take for λ the counting measure, $\lambda(A) = \mathrm{card}\, A$, and this exercise is a small variation on the theme of Theorem 9.8.1.

Exercise 9.8.6 Please read Section A.5 after which everything should look very simple.

Exercise 9.10.2 From (8.47) we have $S(A)\gamma(A^{-1}(p)) = \gamma(p)S(A)$ and thus

$$\begin{aligned}U(a,A)\widehat{D}(\xi)(p) &= \exp(\mathrm{i}(a,p)/\hbar)S(A)\gamma(A^{-1}(p))\xi(A^{-1}(p))\\ &= \gamma(p)U(a,A)(\xi)(p) = \widehat{D}U(a,A)(\xi)(p)\ .\end{aligned}$$

Exercise 9.10.5 The only difference is that for $u \in \mathcal{G}'$ one has $u^\dagger\gamma_0 u = -\|u\|^2$ so that there has to be a minus sign in the definition of the inner product.

Exercise 9.11.2 Starting with the statement $\gamma_\mu S(A) = S(A)\gamma_\nu\kappa(A)^\nu{}_\mu$ we may first raise μ both sides to obtain $\gamma^\mu S(A) = S(A)\gamma_\nu\kappa(A)^{\nu\mu}$. We may then raise the index ν in γ_ν while lowering the index ν in $\kappa(A)^{\nu\mu}$ as is done in Exercise (4.1.5).

Exercise 9.11.3 (a) is integration by parts, using that $(x,p) = x^\mu p_\mu$. (b) The Dirac operator $D := \gamma^\mu\partial_\mu$ satisfies

$$D^2 = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial^\mu\partial_\mu\ . \tag{P.20}$$

If a function f satisfies the Dirac equation, that is $\mathrm{i}\hbar Df = mcf$, then $(\mathrm{i}\hbar)^2 D^2 f = m^2c^2 f$ and hence $\hbar^2\partial^\mu\partial_\mu f + m^2c^2 f = 0$. Every component of f satisfies the Klein-Gordon equation. (c) Follows from the relation $\mathrm{i}\hbar\widehat{\partial_\mu f}(p) = p_\mu \hat{f}(p)$. (d) It follows from (c) that the Fourier transform $\varphi = \hat{f}$ satisfies the equation $\widehat{D}(\varphi) = mc\varphi$ so that by (9.71) we have $(p^2 - m^2c^2)(\varphi)(p) = 0$: this Fourier transform is zero outside $X_m \cup (-X_m)$ (which we already knew since each component is the Fourier transform of the function satisfying the Klein-Gordon equation). It is then a function from $X_m \cup (-X_m) \to \mathbb{C}^4$. When f is real-valued then $\hat{f}(-p) = \hat{f}(p)^*$ so that $\varphi = \hat{f}$. (e) We use

$$\begin{aligned}\hat{f}(\kappa(A^{-1})(p)) &= \int \mathrm{d}^4x \exp(\mathrm{i}(\kappa(A^{-1})(p)), x)/\hbar) f(x) \\ &= \int \mathrm{d}^4x \exp(\mathrm{i}(p,x)/\hbar) f(\kappa(A^{-1}(x)) \ ,\end{aligned}$$

by change of variable $x \to \kappa(A^{-1})(x)$ and Lorentz invariance. (f) These two relations are Fourier transforms of each other.

Exercise 9.12.3 Recalling the formula (9.52), define the representation $\tilde{U}(a,A) := U(Pa, A^{\dagger -1})$. Then the map T given by $T(\varphi)(p) = \varphi(Pp)$ shows that $\tilde{U}(a,A)$ is unitarily equivalent to the representation $U'(a,A)$ of (P.17).

Exercise 9.12.4 We check that $JM(p)^*J^\dagger = M(Pp)$. Consequently $(Jv^*)(Jv^*)^\dagger = JM(p(v))^*J^\dagger = M(Pp(v))$ and thus $p(Jv^*) = Pp(v)$. We then note from Lemma 8.1.4 that $A^{\dagger -1}J = JA^*$ to obtain the formula $TV(Pa, A^{\dagger -1})(v) = V(a,A)T(f)(v)$.

Exercise 9.12.11 Generally speaking, given $Q \in SL^+(2,\mathbb{C})$ and a representation π of $\mathcal{P}^*$ one may define as in (9.12.1) a representation π_Q of $\mathcal{P}^*$ by the formula $\pi_Q(a,A) = \pi(Qa, QAQ^{-1})$. Intuitively, if π is associated to a given particle, the representation π_Q is associated to "the image of the particle through Q". The point is that when $Q^{-1}Q' \in SL(2,\mathbb{C})$ then π_Q and $\pi_{Q'}$ are equivalent. (This applies in particular to the case where Q is parity and Q' mirror symmetry.) This equivalence is a consequence of the following immediate formula: if $W = \pi(0,A)$ for $A \in SL(2,\mathbb{C})$ then $W\pi_Q W^{-1} = \pi_{QA}$.

Exercise 9.13.5 The trick is as always to transport $\varphi(p)$ to $\mathcal{V}_{p^*}$ using $V(D_p)^{-1}$. One then obtains that the representation is equivalent to the representation of $\mathbb{R}^4 \rtimes O(3,1)$ on the space of functions $X_m \to \mathbb{C}^3$ given by the formula

$$U(a,A)(\varphi)(p) = \exp(\mathrm{i}(a,p))(D_p^{-1}AD_{A^{-1}(p)})(\varphi(A^{-1}(p)),$$

where $D_p^{-1}AD_{A^{-1}(p)}$ is viewed as an orthogonal transformation of $\mathbb{C}^3$.

Exercise 10.2.2 Write! That is,

$$W(D)W(C)(e_k) = W(D)\Big(\sum_{\ell\le N} S(C^{-1})_{k,\ell}e_\ell\Big) = \sum_{\ell\le N} S(C^{-1})_{k,\ell}W(D)(e_\ell)$$
$$= \sum_{\ell\le N} S(C^{-1})_{k,\ell}\sum_{\ell'\le N} S(D^{-1})_{\ell,\ell'}e_{\ell'} = \sum_{\ell'\le N}\Big(\sum_{\ell\le N} S(C^{-1})_{k,\ell}S(D^{-1})_{\ell,\ell'}\Big)e_{\ell'}$$
$$= \sum_{\ell'\le N} S(C^{-1}D^{-1})_{k,\ell'}e_{\ell'} = \sum_{\ell'\le N} S((DC)^{-1})_{k,\ell'}e_{\ell'} = W(DC)(e_k)\ .$$

Exercise 10.4.5 Note that Π is simply the orthogonal projection of $\mathbb{C}^N$ on $\mathcal{G}$. Since $\Pi = W^\dagger$, for $x \in \mathbb{C}^N$, $y \in \mathcal{G}$ we have

$$(\Pi S(C)^\dagger x, y) = (x, S(C)Wy) = (x, WS(C^{-1})^\dagger Wy) = (S(C^{-1})\Pi x, y)\ ,$$

where we have used that $S(C)Wy = S(C^{-1})^\dagger Wy = WS(C^{-1})^\dagger Wy$ because $S(C)$ is unitary on $\mathcal{G}$ and since $S(C^{-1})^\dagger Wy \in \mathcal{G}$. This proves (a). The first assertion of (b) follows from (10.21), and the second holds because $S(C)\Pi_p S(C)^\dagger = S(C)S(D_p)\Pi S(D_p)^\dagger S(C)^\dagger = S(CD_p)\Pi S(CD_p)^\dagger$ and $CD_p(p^*) = C(p)$.

To prove (10.22) we compute

$$U(c,C)(\varphi_k) = \exp(\mathrm{i}(c,p)/\hbar)S(C)[\varphi_k(C^{-1}(p)]\ .$$

Now when $\varphi_k = \varphi_k(f) = \Pi(\hat{f}g_k)$ we have $\varphi_k(C^{-1}(p)) = \Pi_{C^{-1}(p)}(\hat{f}(C^{-1}(p))g_k)$ and $S(C)\Pi_{C^{-1}(p)} = \Pi_p S(C^{-1})^\dagger$. Using that $S(C^{-1})^\dagger(g_k) = \sum_{\ell\le N} S(C^{-1})^*_{k,\ell}g_\ell$ we obtain

$$U(c,C)(\varphi_k)(p) = \sum_{\ell\le N} S(C^{-1})^*_{k,\ell}\Pi_p(\exp(\mathrm{i}(c,p)/\hbar)\hat{f}(C^{-1}(p))g_\ell));.$$

Since $\exp(\mathrm{i}(c,p)/\hbar)\hat{f}(C^{-1}(p))$ is the Fourier transform of $V(c,C)f$ this proves the formula 10.22. Applying A to (10.22) implies (10.8).

Exercise 10.5.3 This treatment can be found in [24], Sections 7.3 to 7.5, but it might require dedication to plunge there.

Exercise 10.6.3 (a) We assume that S is irreducible. In Appendix D we prove that S is equivalent to a representation of the type (n_1, n_2) as in Definition 8.3.2. The restriction of S to $G = SU(2)$ is (equivalent to) the representation $\pi_{n_1} \otimes \pi_{n_2}$ of Proposition D.7.6. We also prove in Appendix D that for $j = n_1 + n_2, n_1 + n_2 - 2, \ldots, |n_1 - n_2|$ there is exactly one subspace $\mathcal{G}$ for which the restriction of S to $SU(2)$ and $\mathcal{G}$ is equivalent to π_j. Then $V = V' = \pi_j$.
(b) Taking the conjugate of the matrix relation (10.32) yields $V^*(C) = Z^{-1}S(C)Z$. Since $V^*(C) = V(C^*) = V(J^{-1})V(C)V(J)$ this implies (10.38).
(c) Proposition 10.4.2 implies that $ZV(J^{-1}) = \lambda W$ for some $\lambda \in \mathbb{C}$. Thus $Z = W'^* = \lambda WV(J)$, and (10.36) yields $v(\boldsymbol{p}, q) = \lambda S(D_p)WV(J)(f_q)$. Using (8.12), yields (for a different λ) that for $0 \le q \le j$ we have $v(\boldsymbol{p}, q) = \lambda(-1)^q u(\boldsymbol{p}, j-q)$.

Exercise 10.7.2 We consider only the first of these quantities. For $1 \le k \le N$ the quantities $(u(\boldsymbol{p},q)_k)$ form a column vector $u(\boldsymbol{p},q) = S(D_p)W(f_q)$, as we saw in (10.27). In a similar manner, the quantities $v(\boldsymbol{p},q)_k$ form a column vector $v(\boldsymbol{p},q) = S(D_p)W'(f_q)^*$. Denote by v^T the row vector which is the transpose of a column vector, and by $v^\dagger$ the row vector which is the conjugate-transpose of v. Thus $u(\boldsymbol{p},q)v(\boldsymbol{p},q)^T$ is an $N \times N$ matrix, and the element of this matrix located on row k and column k' is $u(\boldsymbol{p},q)_k v(\boldsymbol{p},q)_{k'}$. Therefore it suffices to show that the matrix

$$\sum_{q\le n} u(\boldsymbol{p},q)v(\boldsymbol{p},q)^T = S(D_p)\sum_{q\le n} W(f_q)W'(f_q)^\dagger S(D_p)^T \tag{P.21}$$

is independent of the choice of D_p and of the orthogonal basis of $\mathcal{H}_0$. That the matrix $M := \sum_{q\le n} W(f_q)W'(f_q)^\dagger$ is independent of the orthonormal basis f_q is rather obvious so we do not detail it. Choose another element D'_p with $D'_p(p^*) = p$, so that $D'_p = D_p C$ where C leaves p^* invariant. It suffices to show that for such C we have $S(C)MS(C)^T = M$. Now

$$V(C) = W^{-1}S(C)W = W'^{-1}S^*(C)W' \tag{P.22}$$

is a unitary transformation of $\mathcal{H}_0$, so that

$$W^{-1}S(C)W(f_q) = W'^{-1}S(C)^*W'(f_q) = \sum_{j\le n} \alpha_q^j f_j \ ,$$

where the α_q^j are the coefficients of an orthonormal matrix. Thus $S(C)W(f_q) = \sum_{j\le n}\alpha_q^j W(f_j)$ whereas in a similar manner $S(C)^*W'(f_q) = \sum_{i\le n}\alpha_q^i W'(f_i)$ and thus, taking adjoints, $W'(f_q)^\dagger S(C)^T = \sum_{i\le n}\alpha_q^{i*}W'(f_i)^\dagger$. The result follows since $\sum_{q\le n}\alpha_q^j\alpha_q^{i*} = \delta_i^j$.

Exercise 10.12.1 We treat only the case of (10.63) since we have already treated the case of (10.64) in a related situation. What this means is that for a test function f we have $\psi^\mu(\partial_\mu f) = 0$. This equation is satisfied separately for ψ^+ and ψ^-. We treat the case of ψ^-, for which (10.52) reads

$$\psi^{-\mu}(f) = \sum_{q\le 3}\int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2c\omega_{\boldsymbol{p}}}(2\pi\hbar)^3}\hat{f}(p)(D_p)^\mu{}_q a^\dagger(\boldsymbol{p},q) \ .$$

Since $\mathrm{i}\hbar\widehat{\partial_\mu f}(p) = p_\mu\hat{f}(p)$ to prove that $\psi^{-\mu}(\partial_\mu f) = 0$ it suffices to prove that $p_\mu(D_p)^\mu{}_q = 0$. Now we have $(p, D_p e_q) = (D_p p^*, D_p e_q) = (p^*, e_q) = 0$ which means exactly $p_\mu(D_p)^\mu{}_q = 0$.

Exercise 10.19.1 Using that $D_p^{\dagger -1} = D_{Pp}$ and letting $C = \sqrt{2mc(mc+p^0)}$ one gets

$$Cu(\boldsymbol{p},1) = \begin{pmatrix} mc + p^0 - p^3 \\ -p^1 - \mathrm{i}p^2 \\ mc + p^0 + p^3 \\ p^1 + \mathrm{i}p^2 \end{pmatrix} \; ; \; Cu(\boldsymbol{p},2) = \begin{pmatrix} -p^1 + \mathrm{i}p^2 \\ mc + p^0 + p^3 \\ p^1 - \mathrm{i}p^2 \\ mc + p^0 - p^3 \end{pmatrix} .$$

Exercise 10.19.2 Parity acts on $\mathcal{H} = L^2(X_m, \mathbb{C}^4, \mathrm{d}\lambda_m)$ by $P'(f)(p) = f(Pp)$. Let us denote by $P_{\mathcal{B}}$ the extension of this operator to the fermion Fock space. The required property is

$$P_{\mathcal{B}} \circ \psi(x) \circ P_{\mathcal{B}} = S(P')\psi(Px) . \tag{P.23}$$

This holds separately for ψ^+ and ψ^-. Making the change of variables $\boldsymbol{p} \to -\boldsymbol{p}$ and using that $(x,p) = (Px, Pp)$ we obtain from (10.110)

$$\psi^+(Px) = \sum_{q \le 2} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3} \frac{c\sqrt{m}}{\sqrt{2\omega_{\boldsymbol{p}}}} \exp(-\mathrm{i}(x,p)/\hbar) u(-\boldsymbol{p},q) a(-\boldsymbol{p},q) .$$

Now, $u(-\boldsymbol{p},q) = S(D_{Pp})f_q$, and $D_{Pp} = D_p^{\dagger -1} = P'D_pP'$. Since $S(P')(f_q) = f_q$ as is apparent from the definition of f_q it holds that $u(-\boldsymbol{p},q) = S(P')S(D_p)(f_q) = S(P')u(\boldsymbol{p},q)$. Thus it suffices to prove that $P_{\mathcal{B}} \circ a(\boldsymbol{p},q) \circ P_{\mathcal{B}} = a(-\boldsymbol{p},q)$. This follows from the relation $P_{\mathcal{B}} A(f) P_{\mathcal{B}} = A(Pf)$ and (10.25).

Exercise 10.20.1 The dimension of the Lagrangian density (10.120) must be $[ml^{-1}t^{-2}]$. The term $mc\bar{\psi}\psi$ of this density shows that ψ should be of dimension $[l^{-1}t^{-1/2}]$. In (10.109) the term $a^\dagger(\boldsymbol{p},q)$ has dimension $[l^{3/2}]$, integration creates a dimension $[l^{-3}]$, and the factor $c\sqrt{m}/\sqrt{\omega_{\boldsymbol{p}}}$ has dimension $[l^{1/2}t^{-1/2}]$, giving the correct dimension. Getting this correct dimension motivated our normalization of the Dirac field.

Exercise 10.21.1 The proof of the last relation is identical to the case of the Dirac field. The first two relations are no longer trivial because now the particle is its own anti-particle, but one may simply reproduce the arguments that were given for the massive Weyl spinor.

Exercise 10.22.1 (a) It follows from (9.85) (or is straightforward to compute) that $J(p^*) = (1,0,0,-1) = Pp^*$. (b) If $B \in SL(2,\mathbb{C})$ and $P'B \in G_0^+$ then $A := P'BQ^{-1} = P'BJ^{-1}P' = (BJ^{-1})^{\dagger -1} \in SL(2,\mathbb{C})$ and also $A \in G_0^+$ since G_0^+ is a group. Thus $A \in SL(2,\mathbb{C}) \cap G_0^+ = G_0$ (c) Follows from Lemma 8.1.4. (d) When $A \in SL(2,\mathbb{C})$ we define $V^+(AQ) = V^+(A)V^+(Q)$. We then check by looking at cases that $V^+(RS) = V^+(R)V^+(S)$ when R, S are of the type A or AQ for $A \in SL(2,\mathbb{C})$.

Exercise 10.22.3 (a) Generally speaking if V is a representation of $SL(2,\mathbb{C})$ in a space $\mathcal{V}$ one may construct a representation W of $SL^+(2,\mathbb{C})$ in $\mathcal{V} \times \mathcal{V}$ using the formula $W(A)(x,y) = (V(A)x, V(A^*)y)$ and $W(P')(x,y) = (V(-J)y, V(J)x)$.

The verification that these formuals make sense is straightfoward using the formula $J^{-1}C^*J = C^{\dagger -1}$ of Lemma 8.1.4. Then $W(Q)(x,y) = (y, V(-I)x)$. Let us chose $\mathcal{V} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $V(A) = A \otimes A$, so that $V(-J) = V(J)$. The tensor $c = (c_{n_1,n_2})_{n_1,n_2\in\{1,2\}}$ such that $c_{2,2} = 1$ and all the other components are zero is such that when $A \in G_0$ we have $V(A)c = \hat{\pi}_2(A)c$ and $V(A^*)c = \hat{\pi}_{-2}(A)c$. The subspace $\mathcal{G}$ of $\mathbb{C}^4 \times \mathbb{C}^4$ consisting of the vectors $(\alpha c, \beta c)$ for $\alpha, \beta \in \mathbb{C}$ has the required properties. (b) There does not exist a vector $c \in \mathbb{C}^4$ such that $S(A)c = \hat{\pi}_2(A)c$ when $A \in G_0$.

Exercise 11.1.2 Since $(B^\dagger B(x), x) = \|B(x)\|^2 \geq 0$ an eigenvalue of $B^\dagger B$ is ≥ 0, so the eigenvalues of $H_0 + \varepsilon H_I$ are $\geq -g^2|\gamma|^2$. Recalling Exercise 2.17.2 an eigenvector of a of eigenvalue $-g\gamma$ is an eigenvector of $H_0 + gH_I$ of eigenvalue $-g^2|\gamma|^2$. For (b), here we have $v_n = e_n$, and $H_I(e_0) = \gamma e_1$, so $(e_0, H_I(e_0) = 0$ and only the term $n = 1$ contributes in (11.13).

Exercise 11.1.3 To minimize the quantity (11.17), if $(v_n)_{n\geq 0}$ is a basis of eigenvectors of H_0 with eigenvalues λ_n (and $\lambda_0 = 0$) let us look for w_1 of the type $\sum_{n\geq 1} \alpha_n v_n$, so that the quantity (11.17) is simply $\sum_{n\geq 0} \lambda_n |\alpha_n|^2 + 2\mathrm{Re}\,\beta_n^* \alpha_n$, where $\beta_n = (v_n, H_I v_0)$. This is minimized by the choice $\alpha_n = -\beta_n/\lambda_n$, which is exactly the choice (11.10). Also the quantity (11.17) has exactly the value (11.12). If you are confused by the fact that here we take $g = 1$ you may try to put the g back.

Exercise 11.2.1 Straightforward:

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle = \exp(\mathrm{i}tH_0)(-H_0)|\psi(t)\rangle + \exp(\mathrm{i}tH_0)H(t)|\psi(t)\rangle$$

Exercise 11.2.3 the hint is immediate. One then applies (11.37) to $t - s$ rather than t, one uses that since $H_I(\theta) = U_0(-\theta)H_I U_0(\theta)$ then $U_0(-s)H_I(\theta)U_0(s) = H_I(-s-\theta)$ and one changes θ_j into $\theta_j + s$.

Exercise 11.4.1 Straightforward.

Exercise 11.5.2 There are three momenta operators, one for each coordinate. Using vector notation to describe the three components simultaneously, the operator is given by $\mathbf{P}|\boldsymbol{m}, (m_{\boldsymbol{\ell}})\rangle = (\boldsymbol{m} + \sum_{\boldsymbol{\ell}} m_{\boldsymbol{\ell}}\boldsymbol{\ell})|\boldsymbol{m}, (m_{\boldsymbol{\ell}})\rangle$. It is obvious that this operator commutes with H_0 since they have a common basis of eigenvectors. We prove that each operator $H_{I,\boldsymbol{k}}$ commutes with $\mathbf{P}$. This is because

$$\langle \boldsymbol{m}, (m_{\boldsymbol{\ell}})|H_{I,\boldsymbol{k}}\mathbf{P}|\boldsymbol{n}, (n_{\boldsymbol{\ell}})\rangle = (\boldsymbol{n} + \sum_{\ell}(n_{\boldsymbol{\ell}}\boldsymbol{\ell}))\langle \boldsymbol{m}, (m_{\boldsymbol{\ell}})|H_{I,\boldsymbol{k}}|\boldsymbol{n}, (n_{\boldsymbol{\ell}})\rangle \;,$$

$$\langle \boldsymbol{m}, (m_{\boldsymbol{\ell}})|\mathbf{P}H_{I,\boldsymbol{k}}|\boldsymbol{n}, (n_{\boldsymbol{\ell}})\rangle = (\boldsymbol{m} + \sum_{\ell}(m_{\boldsymbol{\ell}}\boldsymbol{\ell}))\langle \boldsymbol{m}, (m_{\boldsymbol{\ell}})|H_{I,\boldsymbol{k}}|\boldsymbol{n}, (n_{\boldsymbol{\ell}})\rangle \;,$$

whereas by (11.64)

$$\langle \boldsymbol{m}, (m_{\boldsymbol{\ell}})|H_{I,\boldsymbol{k}}|\boldsymbol{n}, (n_{\boldsymbol{\ell}})\rangle = \langle (m_{\boldsymbol{\ell}})|a_{\boldsymbol{k}}|(n_{\boldsymbol{\ell}})\rangle \int_B \mathrm{d}^3\boldsymbol{x} f_{\boldsymbol{m}}(\boldsymbol{x})^* f_{\boldsymbol{n}}(\boldsymbol{x}) f_{\boldsymbol{k}}(\boldsymbol{x})$$

is not zero only if $\boldsymbol{m} = \boldsymbol{n} + \boldsymbol{k}$ and $m_{\boldsymbol{k}} = n_{\boldsymbol{k}} - 1$ and $m_{\boldsymbol{\ell}} = n_{\boldsymbol{\ell}}$ for $\boldsymbol{\ell} \neq \boldsymbol{k}$. A similar argument proves that each $H^\dagger_{I,\boldsymbol{k}}$ commutes with $\mathbf{P}$.

Exercise 11.5.4 (a) is just another way to write integration in polar coordinates, and (b) is a straightforward application of (a). As for (c), it is the application here of the formula (4.40), here $\int \mathrm{d}r\eta(r)\delta(f(r)) = \eta(r_0)/|f'(r_0)|$ when the equation $f(r) = 0$ has a unique root r_0.

Exercise 11.5.5 When there is no potential one should replace (11.71) by

$$c(\boldsymbol{m},\boldsymbol{n},\boldsymbol{k},t) = g^2\theta(\boldsymbol{k}^2)^2 C(\boldsymbol{m},\boldsymbol{n},\boldsymbol{k})^2 f(E_{\boldsymbol{m}} - E_{\boldsymbol{n}} - \omega_{\boldsymbol{k}}, t) \; , \tag{P.24}$$

where $E_{\boldsymbol{m}} = \boldsymbol{m}^2/2M$, and where $C(\boldsymbol{m},\boldsymbol{n},\boldsymbol{k}) = 1/L^{3/2}$ if $\boldsymbol{m} = \boldsymbol{k} + \boldsymbol{n}$ and is zero otherwise. To compute the probability of transition from the state of momentum $\boldsymbol{m}$ to any other state one has to perform the summation over $\boldsymbol{n}$, or equivalently over $\boldsymbol{k}$. Approximating this summation by an integral yields the expression

$$g^2 \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} f\Big(\frac{\boldsymbol{m}^2}{2M} - \frac{(\boldsymbol{m}-\boldsymbol{p})^2}{2M} - \omega_{\boldsymbol{p}}, t\Big) \; .$$

and the result by the usual approximation of $f(x,t)$.

Exercise 11.5.8 I should not waste your chance for a refund of this book, should I?

Exercise 12.2.5 Consider continuous functions φ_n with compact support with $\|\varphi_n\| \le 2^{-n}$ and $\varphi_n(0) \ge 2^{n+1}$. Thus for any sequence (t_k) the function $\varphi = \sum_{k\ge1} U_0(-t_k)(\varphi_k) \in L^2$. Now

$$|U_0(t_n)\varphi(0)| \ge \varphi_n(0) - \sum_{k\neq n} |U_0(t_n - t_k)\varphi_k(0)|; .$$

and one may recursively choose the points t_n so that for $n \neq k$ one has $|U_0(t_n - t_k)\varphi_k(0)| \le 2^{-\max(k,n)}$. For this we use that when ψ is continuous with compact support $U(t)\psi$ converges uniformly to zero, so we simply choose t_n large enough that for $k < n$ we have both $|U(t_n - t_k)\varphi_k(0)| \le 2^{-n}$ and $|U(t_k - t_n)\varphi_n(0)| \le 2^{-n}$.

Exercise 12.2.7 According to (11.23), in the interaction picture the state ψ evolves at time t into $U_0(t)^{-1}U(t)\psi$, so that the state $\psi = U(t)^{-1}U_0(t)\varphi$ evolves at time t into φ. Taking $t = -\infty$ and $\varphi = |\xi\rangle$, the state $|\xi\rangle_{\mathrm{in}} = U(-\infty)^{-1}U_0(-\infty)|\xi\rangle$ evolves at time $t = -\infty$ into $|\xi\rangle$.

Exercise 12.4.2 In the case where there is no scattering, $\Xi \equiv 0$, it follows from (12.32) that $\Phi(|\varphi\rangle_u) = \int \mathrm{d}^3\boldsymbol{p}\theta(\boldsymbol{p})|\langle\boldsymbol{p}|\varphi\rangle|^2/(2\pi\hbar)^3$ so that this quantity must be zero in order for the quantity $\int_{\|u\|\le R} \mathrm{d}^2u\Phi(|\varphi\rangle_u)$ to stay bounded as $R \to \infty$.

Exercise 13.12.1 Indeed we then have $\boldsymbol{p}_1 = -\boldsymbol{p}_2$ so that $p_1^0 = p_2^0$. Since $0 = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_3 + \boldsymbol{p}_4$ we also have $\boldsymbol{p}_3 = -\boldsymbol{p}_4$ so that $p_3^0 = p_4^0$. Since $p_1 + p_2 = p_3 + p_4$ we then have $p_1^0 = p_2^9 = p_3^0 = p_4^0$, from which the claim follows.

Exercise 13.3.1 We have $S^\dagger = \sum_{n\geq 0} S_n^\dagger$ where

$$S_n^\dagger = \frac{(\mathrm{i}g)^n}{n!}\int \mathrm{d}^4x_1 \ldots \int \mathrm{d}^4x_n \bar{\mathcal{T}}\mathcal{H}(x_1)\cdots\mathcal{H}(x_n)\ , \tag{P.25}$$

and where $\bar{\mathcal{T}}$ denotes "reverse time ordering". Computing $S^\dagger S$ we obtain $1+\sum_{n\geq 1} g^n A_n$ where obviously $A_1 = 0$ and

$$2A_2 = \int \mathrm{d}^4x_1\mathrm{d}^4x_2(\mathcal{H}(x_1)\mathcal{H}(x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) - \mathcal{T}\mathcal{H}(x_1)\mathcal{H}(x_2) - \bar{\mathcal{T}}\mathcal{H}(x_1)\mathcal{H}(x_2))$$

is zero because the integrand is zero. To prove that A_3 is zero, denoting $\mathcal{H}(x_i)$ by h_i one reduces to the identity $\mathcal{T}(h_1h_2h_3) - h_1\mathcal{T}(h_2h_3) - h_2\mathcal{T}(h_1h_3) - h_3\mathcal{T}(h_1h_2) + \bar{\mathcal{T}}(h_2h_3)h_1 + \bar{\mathcal{T}}(h_1h_3)h_2 + \bar{\mathcal{T}}(h_1h_2)h_3 - \bar{\mathcal{T}}(h_1h_2h_3) = 0$, which is proved by writing what it means when $x_1 \geq x_2 \geq x_3$.

Exercise 13.5.1 Since both sides are anti-symmetric tensors, it suffices to consider the case $k = 1, k' = 2$ and the right-hand side is the determinant of C^{-1} which is 1.

Exercise 13.9.1 (a) We have

$$\mathrm{Re}\, B = 2\int_{\{0\leq\theta_1\leq\theta_2\leq t\}} \mathrm{d}\theta_1\mathrm{d}\theta_2 \mathrm{Re}\,(\alpha(\theta_2)^*\alpha(\theta_1))\ ,$$

and since $\mathrm{Re}\,(\alpha(\theta_2)^*\alpha(\theta_1)) = \mathrm{Re}\,(\alpha(\theta_1)^*\alpha(\theta_2))$ we get

$$\mathrm{Re}\, B = \int_{\{0\leq\theta_1,\theta_2\leq t\}} \mathrm{d}\theta_1\mathrm{d}\theta_2 \mathrm{Re}\,(\alpha(\theta_2)^*\alpha(\theta_1)) = |A|^2\ .$$

For (b), $U(t)|0\rangle$ is of the type $U(t)|0\rangle = \sum_{k\geq 0}\lambda_k e_k$. Then $\langle 0|a^k U(t)|0\rangle = \sqrt{k!}\lambda_k$ and since $|\langle 0|a^k U(t)|0\rangle|^2 = |\langle 0|a^k V(t)|0\rangle|^2$ by (13.53) we get $|\lambda_k|^2 = |A|^{2k}\exp(-\mathrm{Re}\, B)/k!$ which sums to 1 by (a), and $|\lambda_k|^2$ is of the probability that $U(t)$ has k quanta of oscillation.

Exercise 13.10.5 Just compute the integral:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t \exp(-|t|a + \mathrm{i}t\omega) &= \int_{-\infty}^{0} \mathrm{d}t \exp(ta + \mathrm{i}t\omega) + \int_{0}^{\infty} \mathrm{d}t \exp(-ta + \mathrm{i}t\omega)\\ &= \frac{1}{a+\mathrm{i}\omega} - \frac{1}{-a+\mathrm{i}\omega} = \frac{2a}{\omega^2+a^2}\ ,\end{aligned} \tag{P.26}$$

and (13.64) is just application of the inverse Fourier transform to this relation. Denoting by $\omega_{\boldsymbol{p},\varepsilon}$ the root of the equation $z^2 = \boldsymbol{p}^2 + m^2 - \mathrm{i}\varepsilon$ with negative imaginary part, one uses (13.64) with $a = \mathrm{i}\omega_{\boldsymbol{p},\varepsilon}$ to obtain (13.62).

Exercise 13.10.6 (a) You would write

$$\Delta_F(x)^2 = \lim_{\varepsilon\to 0^+}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4p'}{(2\pi)^4}\frac{\exp(-\mathrm{i}(x, p+p'))}{(-p^2+m^2-\mathrm{i}\varepsilon)(-p'^2+m^2-\mathrm{i}\varepsilon)}\ ,$$

and integrating against a test function f

$$\int \mathrm{d}^4x f(x)\Delta_F(x)^2 = \lim_{\varepsilon\to 0}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4p'}{(2\pi)^4}\frac{\hat{f}(-p-p')}{(-p^2+m^2-\mathrm{i}\varepsilon)(-p'^2+m^2-\mathrm{i}\varepsilon)}\ .$$

The integral is however not convergent in general. To get convinced of that, you may integrate in p' at given p. Assuming that $\hat{f} > 0$ in a neighborhood of 0 you get an integral of order at least $1/\|p\|^4$ (where $\|p\|$ denotes the Euclidean norm) and this is not an integrable function of p.

(b) Formal manipulations yield the definition $\Delta_0^2(f) = \int \mathrm{d}\lambda_m(p)\mathrm{d}\lambda_m(p')\hat{f}(p+p')$. This integral is well defined because $\lambda_m^2(\{(p,p')\ ;\ p^0+p'^0 \le a\})$ grows polynomially in a and $\hat{f}$ decreases fast at infinity.

Exercise 13.13.3 (a) is a linguistic issue. Renumbering the set of internal vertices defines a permutation of this set (sending a point to the point which has the same order in the new ordering) and conversely. The rules we state about the permutation exactly amount to saying the corresponding renumbering of the internal vertices does not change the diagram. (b) If an internal vertex i is connected to an external vertex v then $\sigma(i)$ has to be connected to v. But there is unique internal vertex connected to a given external vertex, so that $\sigma(i) = i$.

Exercise 13.13.5 Let us think of the symmetry as a permutation of the set of vertices (internal and external) which leaves the set of external vertices fixed and does not change the diagram. The set of vertices which are fixed is not empty because it contains the external vertices, and there is at least one such vertex. If a vertex is connected to a vertex f which is fixed it must be fixed because there only one line of a given type out of f. Since we assume that the whole diagram is connected, the set of points which are fixed must be the whole diagram. Note on the other hand than in the examples of Figure 13.6 there exist symmetries which do not fix any vertex.

Exercise 13.14.1 (a) We write $\varphi(x)$ as the sum $\varphi(x) = \varphi^+(x) + \varphi^-(x)$ of an annihilation and a creation part, so that say $\langle 0|a(p)\varphi(x)|0\rangle = \langle 0|a(p)\varphi^-(x)|0\rangle$, whereas $\langle 0|\varphi^-(x)a^\dagger(p)|0\rangle = 0$. Then $:\!\varphi(x)^4\!:= \varphi^-(x)^4 + 4\varphi^-(x)^3\varphi^+(x) + 6\varphi^-(x)^2\varphi^+(x)^2 + 4\varphi^-(x)\varphi^+(x)^3 + \varphi^+(x)^4$. We simply expand before we can apply Lemma 13.8.1. To understand why there are fewer terms and (b) is true let us number $\varphi_j(x)$, $1 \le j \le 4$ the four copies of $\varphi(x)$ in the product $\varphi(x)^4$. The lines between the internal vertex corresponding to x and itself occur because of contractions such as $\langle 0|\varphi_1(x)\varphi_2(x)|0\rangle$, or after expansion in creation and annihilation part, because of contractions of the type $\langle 0|\varphi_1^+(x)\varphi_2^-(x)|0\rangle$. However, these terms do not occur when we replace $\varphi(x)^4$ by $:\!\varphi(x)^4\!:$ because the normal ordering replaces $\varphi_1^+(x)\varphi_2^-(x)$ by $\varphi_2^-(x)\varphi_1^+(x)$.

Exercise 13.14.2 (a) is a special case of (b). Let us try more generally for a test function f to make sense of the operator $W := \int \mathrm{d}^4x f(x)\ :\!\varphi(x)^4\!:$, as an operator on a certain subspace of the Boson Fock space, which we define now. Let us say that a function ξ on X_m^n is of fast decrease if for each k the function $(\sum_{i\le n} p_i^0)^k\xi$ is bounded. Let $\mathcal{H}_n^{\text{fast}}$ be the set of symmetric functions of fast decrease on X_m^n. We will try to define our operators on the algebraic sum of the spaces $\mathcal{H}_n^{\text{fast}}$ (a nice subspace of the boson Fock space.) Computations are easier using the formula (5.42). A typical term in $:\varphi(x)^4:$ is

$$\iiiint \mathrm{d}\lambda_m(p_1)\mathrm{d}\lambda_m(p_2)\mathrm{d}\lambda_m(p_3)\mathrm{d}\lambda_m(p_4) \\ \exp(\mathrm{i}(x, p_1+p_2-p_3-p_4))a^\dagger(p_1)a^\dagger(p_2)a(p_3)a(p_4)$$

and a typical term in W is

$$W_0 = \iiiint \mathrm{d}\lambda_m(p_1)\mathrm{d}\lambda_m(p_2)\mathrm{d}\lambda_m(p_3)\mathrm{d}\lambda_m(p_4) \\ \hat{f}(p_1+p_2-p_3-p_4)a^\dagger(p_1)a^\dagger(p_2)a(p_3)a(p_4)\ .$$

Arguing as in Exercise 3.7.1 (and crossing our fingers because the function $\hat{f}$ is not a test function!) and not being concerned with the numerical factor, for $\xi \in \mathcal{H}_n^{\text{fast}}$, we should define $W_0\xi$ as being proportional to the symmetrization of the following function

$$\eta(p_1,\ldots,p_n) = \iint \mathrm{d}\lambda_m(p_1')\mathrm{d}\lambda_m(p_2')\hat{f}(p_1+p_2-p_1'-p_2')\xi(p_1',p_2',p_3,\ldots,p_n)$$

with respect to the variables $p_1,\ldots,p_n$. To show that this function decreases fast, one simply splits the integral in the regions where $p_1'^0+p_2'^0 \le (p_1^0+p_2^0)/2$ and its complement and one uses easy bounds. The same computation even makes sense when $f\equiv 1$. (c) If f is a test function on $\mathbb{R}^3$, proceeding as above $\int \mathrm{d}\boldsymbol{x} f(\boldsymbol{x})\ :\varphi_0(\boldsymbol{x})^4:|0\rangle$ should be the function $\hat{f}(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4)$ on X^4. But such a function is not square-integrable, and it seems very difficult to make sense of $H_I(0)$. However, despite the fact that we are not in the situation of Section 13.3, we are able to compute the S matrix in ϕ^4 theory, so that this theory makes some sense after all.

Exercise 13.14.3 Let us write $\varphi(x)=\varphi^+(x)+\varphi^-(x)$, the annihilation and creation parts of φ. Pretending that $[\varphi^-(x),\varphi^+(x)]=\Delta 1$ for a certain number Δ, the trick is to write $:\varphi(x)^4:$ as a linear combination of $\varphi(x)^4, \varphi(x)^2\Delta$ and $\Delta^2\mathbf{1}$, which makes the desired result formally obvious. You may try first to write first $:\varphi(x)^2:$ and $:\varphi(x)^3:$ in this manner.

Exercise 13.15.1 Writing $\mathcal{C} := \langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)|0\rangle^2$, these are

$$\langle 0|\varphi(x_1)a^\dagger(p_1)|0\rangle\langle 0|\varphi(x_1)a^\dagger(p_2)|0\rangle\mathcal{C}^2\langle 0|a(p_3)\varphi(x_2)|0\rangle\langle 0|a(p_4)\varphi(x_2)|0\rangle\ ,$$

$$\langle 0|\varphi(x_1)a^\dagger(p_1)|0\rangle\langle 0|\varphi(x_2)a^\dagger(p_2)|0\rangle\mathcal{C}^2\langle 0|a(p_3)\varphi(x_1)|0\rangle\langle 0|a(p_4)\varphi(x_2)|0\rangle\ ,$$

$$\langle 0|\varphi(x_1)a^\dagger(p_1)|0\rangle\langle 0|\varphi(x_2)a^\dagger(p_2)|0\rangle\mathcal{C}^2\langle 0|a(p_4)\varphi(x_1)|0\rangle\langle 0|a(p_3)\varphi(x_2)|0\rangle\ .$$

Exercise 13.15.2 For Exercise 13.13.1: The number of different contraction diagrams one obtains when labeling the internal vertices of a Feynman diagram all possible ways and the lines out the internal vertices all possible ways is of the form $n!(4!)^n/S$ where S is an integer called the *symmetry factor* of the diagram. For Lemma 13.13.2: The symmetry factor S of a contraction diagram is the number of ways one may relabel the internal vertices and relabel the lines out the internal vertices without changing the diagram.

Exercise 13.18.1 Let us do this the easy way. We compute the integral using the residue formula and the contour of Figure 13.2. The existence of the limit is obvious, as each term has a limit and denominators do not approach zero. The quantity $A(\boldsymbol{p})$ is the sum of two terms of the type $1/g(\boldsymbol{p})$. In one of them the function $g(\boldsymbol{p})$ is a constant times the quantity

$$\sqrt{\boldsymbol{p}^2+m^2}\Big(-(\sqrt{\boldsymbol{p}^2+m^2}-w_0)^2+(\boldsymbol{p}-\boldsymbol{w})^2+m^2\Big)\ .$$

A function of the type $1/g(\boldsymbol{p})$, where g is smooth and the set $S=\{g=0\}$ is a non-empty two-dimensional surface is never integrable, exactly for the same reason that in dimension 1 a function of the type $1/f(x)$ where f is smooth and has at least one zero is never integrable. The volume of the set where $|g(\boldsymbol{p})|\le\alpha$ is proportional to α (the volume of a shell of thickness about α around the surface S). It does take some effort to show that here the set $\{g=0\}$ is typically a two-dimensional surface, and you are on your own for that.

Exercise 13.18.3 (a) is straightforward. It does not make sense to set $\varepsilon=0$ because the left-hand side becomes a massive $+\infty$, and it does not make sense either to replace ε by zero in the integral in (13.124).

Exercise 13.18.6 Let us consider w and w' and fix R with $R^2\ge\max(w^2,w'^2)$. Under (13.129) the relation (13.134) implies

$$|U(w,\theta)-U(w',\theta)-\mathrm{i}(U_1(w,R)-U_1(w',R))|\le|U_2(w,R,\theta)-U_2(w',R,\theta)|\ . \quad \text{(P.27)}$$

According to (13.135), to conclude the proof it suffices to show that

$$\lim_{R\to\infty}\sup_\theta|U_2(w,R,\theta)-U_2(w',R,\theta)|=0\ , \quad \text{(P.28)}$$

where the supremum is over all values of θ for which $\theta(\boldsymbol{p})=1$ when $\boldsymbol{p}^2\le R^2$. Making the change of variable $p\to p-uw$ we obtain

$$U_2(w,R,\theta)=\int_0^1\mathrm{d}u\int_{\|p-uw\|\ge R}\frac{\theta(\boldsymbol{p})\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(\|p-uw\|^2-u(1-u)w^2+m^2)^2}\ , \quad \text{(P.29)}$$

so that

$$U_2(w,R,\theta)=U_3(w,R,\theta)+U_4(w,R,\theta)\ , \quad \text{(P.30)}$$

where

$$U_3(w,R,\theta)=\int_0^1 \mathrm{d}u\int_{\|p\|\geq R}\frac{\theta(\boldsymbol{p})\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(\|p-uw\|^2-u(1-u)w^2+m^2)^2}\ ,$$

and where (P.30) defines $U_4(w,R,\theta)$. Since $0\leq\theta(\boldsymbol{p})\leq 1$, considering the symmetric difference $\Delta(w,u,R)$ between the sets $\{\|p\|\geq R\}$ and $\{\|p-uw\|\geq R\}$, we have

$$|U_4(w,R,\theta)|\leq U_5(w,R):=\int_0^1 \mathrm{d}u\int_{\Delta(w,u,R)}\frac{1}{(\|p-uw\|^2-u(1-u)w^2+m^2)^2}\ .$$

Then $U_5(w,R)\to 0$ as $R\to\infty$ since for large R we integrate a quantity of order R^{-4} on a domain of volume of order R^3. Therefore since $0\leq\theta(\boldsymbol{p})\leq 1$ we have

$$|U_2(w,R,\theta)-U_2(w',R,\theta)|\leq\int_0^1 \mathrm{d}u\int_{\|p\|\geq R}\frac{\mathrm{d}^4p}{(2\pi)^4}H(w,w',p,u)+\mathcal{R}(R)\ ,\quad\text{(P.31)}$$

where $\mathcal{R}(R)=U_5(w,R)+U_5(w',R)\to 0$ and the function $H(w,w',p,u)$ is given by

$$\left|\frac{1}{(\|p-uw\|^2-u(1-u)w^2+m^2)^2}-\frac{1}{(\|p-uw'\|^2-u(1-u)w'^2+m^2)^2}\right|$$

and is integrable. Letting $R\to\infty$ in (P.31) then yields the desired result.

Exercise 13.23.2 You have to replace (13.175) by

$$\frac{1}{2}(U(w,\theta)-U(0,\theta))\Big(\frac{1}{2}U(w,\theta)+\frac{5}{2}U(0,\theta)\Big)\ ,$$

when $w=p_1-p_3$ for the s-group and $w=p_1-p_4$ for the t-group.

Exercise 14.3.4 This Hamiltonian is the natural extension to the Fock space of the operator "multiplication by cp^0 on $L^2(X_m,\mathrm{d}\lambda_m)$. The Fock space is the Hilbert sum of the spaces $H_{n,s}$, each of which is invariant under the Hamiltonian, so that is suffices to prove that for $n\geq 1$ the extension V_n of the operator "multiplication by cp^0" to $H_{n,s}$ does not have any eigenvector. The space $H_{n,s}$ identifies the subspace of $L^2(X_m^n,\mathrm{d}\lambda_m^{\otimes n})$ consisting of symmetric functions, and V_n is the operator "multiplication by f" where f is the function $c\sum_{k\leq n}p_k^0$ of the point $(p_k)_{k\leq n}$ of X_m^n. An eignenvector of V_n must be supported by the set where f is constant, but such sets are of measure zero.

Exercise 14.5.1 (a) We may as well assume $a_0=1$. The result follows from the formal identity $(1+C)^{-1}=\sum_{n\geq 0}(-C)^n$ used for $C=\sum_{k\geq 1}a_kg^k$. (b) To express a formal series $\sum_{n\geq 0}b_ng^n$ as a formal series in g' we simply substitute the expression $g=g'+\sum_{n\geq 2}a_n(g')^n$ in each term and we expand. This makes sense because there are only finitely many terms which contain a given power of g. To express g' as a formal series in g we look for an expression $g'=g+\sum_{n\geq 2}c_ng^n$. We substitute this in the expression $g=g'+\sum_{n\geq 2}a_n(g')^n$ and this gives us equations from which we recursively compute the coefficients c_n, for example $c_2=-a_2$ and $c_3=-a_3-2(a_2)^2$.

Exercise 14.10.1 We look for θ of the type $\theta = m^2 + g^2\Gamma_2(m^2) + g^4 B + O(g^6)$. Thus $\Gamma(\theta) = g^2\Gamma_2(\theta) + g^4\Gamma_4(\theta) + O(g^6)$. Since $\Gamma_4(\theta) = \Gamma_4(m^2) + O(g^2)$ and $\Gamma_2(\theta) = \Gamma_2(m^2) + g^2\Gamma_2(m^2)\Gamma_2'(m^2) + O(g^4)$ we obtain $\Gamma(\theta) = g^2\Gamma_2(m^2) + g^4(\Gamma_2(m^2)\Gamma_2'(m^2) + \Gamma_4(m^2)) + O(g^6)$, and $B = \Gamma_2(m^2)\Gamma_2'(m^2) + \Gamma_4(m^2)$ from (14.97).

Exercise 14.10.2 If a diagram has n internal vertices, i internal lines and e external lines then $3n = 2i + e$. To see that, think of each internal vertex as providing three slots, each of which has to be filled by the end of a line. The end of the e external lines fill e such slots, and the internal lines each fill two of the slots, one with each of their ends. Thus when e is even, so is n.

Exercise 14.12.3 That $\mu = m$ follows from (14.101) (hoping of course that nothing goes wrong). The second statement follows (14.124) using l'Hospital rule.

Exercise 15.4.12 According to (15.41) and (15.38) we have $\operatorname{card}\mathcal{E}_1 + 2\operatorname{card}(\mathcal{E}_2 \setminus \mathcal{E}') = b \le 4$. Since the whole diagram is connected we must have $\operatorname{card}\mathcal{E}_1 > 0$. Since b is even $\operatorname{card}\mathcal{E}_1$ is even, so that $\operatorname{card}\mathcal{E}_1 \ge 2$. Thus $\operatorname{card}(\mathcal{E}_2 \setminus \mathcal{E}') \le 1$: either the sub-diagram is a subgraph or it is obtained from a subgraph by removing a single edge. The only possible cases are (a) $\operatorname{card}\mathcal{E}_1 = 2$ and $\operatorname{card}(\mathcal{E}_2 \setminus \mathcal{E}') = 0$: the sub-diagram α is a biped and $d(\alpha) = 2$. (b) $\operatorname{card}\mathcal{E}_1 = 2$ and $\operatorname{card}(\mathcal{E}_2 \setminus \mathcal{E}') = 1$: the sub-diagram α has been obtained from a biped by removing a single edge, and $d(\alpha) = 0$. (c) $\operatorname{card}\mathcal{E}_1 = 4$ and $\operatorname{card}(\mathcal{E}_2 \setminus \mathcal{E}') = 0$: the sub-diagram α is a quadruped and $d(\alpha) = 0$.

Exercise 15.5.2 This space has a base formed by the elements $(e_1 - e_k)$ for $2 \le k \le n$.

Exercise 15.7.3 Fix an arbitrary point $v_0 \in \mathcal{V}$ and for $\bar{w} \in (\mathbb{R}^{1,3})^{\mathcal{V}\setminus\{v_0\}}$ define $S(\bar{w}) \in (\mathbb{R}^{1,3})^{\mathcal{V}}$ by $S(\bar{w})_v = \bar{w}_v$ for $v \ne v_0$ and $S(\bar{w})_{v_0} = -\sum_{v \ne v_0} \bar{w}_v$ so that $S(\bar{w}) \in \mathcal{N}$. Integrating in w_{v_0} first we obtain

$$\int \frac{\mathrm{d}^{4m} w}{(2\pi)^{4m}} (2\pi)^4 \delta^{(4)}\Big(\sum_{v\in\mathcal{V}} w_v\Big)\eta(w) = \int \frac{\mathrm{d}^{4(m-1)}\bar{w}}{(2\pi)^{4(m-1)}} \eta(S(\bar{w})) \,. \tag{P.32}$$

The integral on the right is with respect to a translation invariant measure, the image of the translation invariant measure on $(\mathbb{R}^{1,3})^{\mathcal{V}\setminus\{v_0\}}$ under the linear map S.

Exercise 16.3.2 It should be obvious that $\ker\mathcal{L}$ consists of the vectors of the type (ℓ, ℓ, ℓ) and that the projection of x on $\ker\mathcal{L}$ is obtained for the value of ℓ given.

Exercise 16.3.3 With obvious notation we have $\mathcal{L}(Ax) = A\mathcal{L}(x)$ so that $\ker\mathcal{L}$ is invariant by A. Obviously A preserves the dot product on $(\mathbb{R}^{1,3})^{\mathcal{E}}$ so that $A\mathcal{Q} = \mathcal{Q}$. The equality $x = \mathcal{I}(x) + \mathcal{T}(x)$ implies $Ax = A\mathcal{I}(x) + A\mathcal{T}(x)$, and since $A\mathcal{I}(x) \in \ker\mathcal{L}$ and $A\mathcal{T}(x) \in \mathcal{Q}$ we have $A\mathcal{I}(x) = \mathcal{I}(Ax)$ and $\mathcal{T}(Ax) = A\mathcal{T}(x)$. Next you have to convince yourself that for a function H on $\mathcal{Q}$ the Taylor polynomial of order d at $q = 0$ of the function $q \mapsto H(Aq)$ is $H^d(Aq)$ where H^d is the Taylor polynomial of order d at $q = 0$ of H. Consequently (using the same notation as below (16.11)) the

Taylor polynomial of order d at $q = 0$ of the function $q \mapsto F_A(z+q) = F(Az+Aq)$ is $G_d(Az, Aq)$. Thus $T^d F_A(x) = G_d(A\mathcal{I}(x), A\mathcal{T}(x)) = G_d(\mathcal{I}(Ax), \mathcal{T}(Ax)) = T^d F(Ax)$ which is the required equality.

Exercise 16.5.6 The forests are $\{\gamma\}$; $\{\gamma_1, \gamma\}$; $\{\gamma_2, \gamma\}$; $\{\gamma_3, \gamma\}$; $\{\gamma_4, \gamma\}$; $\{\gamma_1, \gamma_2, \gamma\}$; $\{\gamma_1, \gamma_3, \gamma\}$; $\{\gamma_2, \gamma_4, \gamma\}$.

Exercise 16.6.2 (a) Use (16.12) and Exercise 16.3.2. (b) Proceeding in a similar manner for the diagram β we obtain $\mathcal{F}_2 F = -f(p/2 - q)f(p/2 + q)f(q/3 - \ell)^3$. For example if ℓ is fixed, the quantity $\mathcal{F}_2 F$ is integrable in q by itself whereas the quantity $F + \mathcal{F}_1 F$ is integrable in q. (In fact, $\mathcal{F}_1 F$ has been designed for this purpose.)

Exercise 16.6.3 Given $p, r, q, \ell \in \mathbb{R}^{1,5}$ consider $x(p, r, \ell, q) \in (\mathbb{R}^{1,5})^{\mathcal{E}_\alpha}$ given by the parameterization of Figure 16.6. Then obviously $x(0, r, 0, q) \in \ker \mathcal{L}_\alpha$. Since by (15.34) we have $\dim \ker \mathcal{L}_\alpha = 6(6 - 5 + 1) = 12$, all the elements of $\ker \mathcal{L}_\alpha$ are of the type $x(0, r, 0, q)$. On the other hand, it is straightforward to check that $x(p, 0, \ell, 0)$ and $x(0, r, 0, q)$ are orthogonal. This means that $x(p, 0, \ell, 0) \in \ker \mathcal{L}_\alpha^\perp$ so that $x(0, r, 0, q) = \mathcal{I}_\alpha(x(p, r, \ell, q))$. This should make the formula for $\mathcal{F}_1 F$ obvious, the flows on the edges of α are replaced by the internal flows. What happens here is that even though the sum $F + \mathcal{F}_2 F$ does not have a divergence in the sub-diagram β (as the term $\mathcal{F}_2 F$ is really designed to remove this divergence), the term $\mathcal{F}_1 F$ bring in a new divergence in β.

Exercise 16.6.4 To improve convergence we replace F by $(1 - T^{d(\gamma)})F$, so we should also do this for subdivergences. If you have disjoint subdiagrams $\gamma_1, \ldots, \gamma_n$, you should use that process "on each γ_i". Using the identity $\prod_{i \le n}(1 - x_i) \sum_{I \subset \{1,\ldots,n\}} = \prod_{i \in I}(-x_i)$ you see that a good idea is to add a compensating term $\sum_{I \subset \{1,\ldots,\}} \mathcal{F}_I F$, where $\mathcal{F}_I$ is the forest containing the diagrams γ and γ_i for $i \in I$. The reason you assume that $\gamma_1, \ldots, \gamma_n$ are disjoint is simply that it is far from obvious to see what to do for subdiagrams which are not disjoint. Once you see that you must add such a compensating term to F, arguing that you must apply the same procedure "on each γ_i" it does not seem to require much imagination to invent the forest formula. The previous argument shows that it seems sort of necessary to have a chance of success to add all the terms of the forest formula. Why it is sufficient to add these terms is a different matter.

Exercise 16.7.1 The diagram obtained by contracting each of the connected components cannot contain a loop because then the Feynman diagram would contain a loop such that removing any edge of this loop would disconnect the diagram, which is absurd. Next assume if possible that one of the connected components A of the remaining diagram is not 1-PI. Then it contains an edge e such that removing e disconnects this component into pieces B and C. We prove that removing e disconnects the original diagram which contradicts the fact that e is an edge of A. We proceed by contradiction. If removing e does not disconnect the original diagram, this diagram contains a path linking a point of B to a point of C and this path must contain edges which are not edges of A. These edges form a loop in the diagram obtained by contracting the connected components, which we showed is impossible.

Exercise 17.6.2 Let us stress that Lorentz invariance is really built in at every step of the theory, which is why we did not insist more on it. Denoting by $I(p_1)$ the quantity (17.14) we have to show that for any Lorentz transformation A we have $I(p_1) = I(Ap_1)$. For vector $x = (x_e)_{e\in\mathcal{E}} \in (\mathbb{R}^{1,5})^{\mathcal{E}}$ we denote Ax the element $(Ax_e)_{e\in\mathcal{E}}$. Since the components $\chi(p_1)_e$ are all multiples of p_1 (because of the general fact that they are linear combinations of the external momenta) we have $\chi(Ap_1) = A\chi(p_1)$. Thus $T^2\mathcal{F}(Ax + \chi(Ap_1)) = T^2\mathcal{F}(A(x + \chi(p_1)))$. The first thing we have to check is that $T^2F(Ay) = T^2F(y)$. It is certainly true that $F(Ay) = F(y)$ because the propagator f is Lorentz invariant, and Exercise 16.3.3 shows that the operation T^2 preserves Lorentz invariance. Next you have to convince yourself that the measure $\mathrm{d}\mu_{\mathcal{L}}$ is invariant by the map $a \mapsto Ax$ (so that then the equality $I(p_1) = I(Ap_1)$ follows by this change of variable), which requires re-examining the definition of this measure, and ultimately relies on the fact that the volume measure on $\mathbb{R}^{1,5}$ is invariant by Lorentz transformations,

Exercise 17.6.3 The value of this diagram is

$$(2\pi)^6\delta^{(6)}(p_1+p_2+p_3)(-\mathrm{i}g)^3\int\frac{\mathrm{d}^6k}{(2\pi)^6}f(k+q_1)f(k+q_2)f(k+q_3) \tag{P.33}$$

where q_1, q_2, q_3 are certain linear combinations of the external momenta p_1, p_2, p_3. The way to enforce the first condition (17.7) at order g^3 is to define at this stage the counter-term D by $D = -(-\mathrm{i}g)^3\int\frac{\mathrm{d}^6k}{(2\pi)^6}f(k)^3$, which at this order is exactly the value following form (17.15). (Please note that in (17.15) there is exactly one term of order g^3, corresponding to the tripod α_0 with three internal vertices and the unique possible forest on α_0, consting of α_0 itself.

Exercise 17.6.4 Keeping again the cutoff implicit, the relevant integral is

$$U(p) := (-\mathrm{i}g)^2\int\frac{\mathrm{d}^6k}{(2\pi)^6}f(k+p/2)f(k-p/2)\ . \tag{P.34}$$

Since we pretend that our cutoff is Lorentz invariant, the quantity $U(p)$ depends only on p^2: it is of the type $Y(p^2)$. The way to enforce the second part of (17.7) at order g^2 is to set $B = -Y'(0)$ and $C = -Y(0)$. Rather remarkably, the contribution of the single vertex $\otimes$ then cancels the divergence of β_0 at order 2, because the quantity $H(p^2) := Y(p^2) - p^2Y'(0) - Y(0)$ is given by a convergent integral. A simple way to see it is to proceed as in the BPHZ method, to replace the integrand $V(k,p) = f(k+p/2)f(f+p/2)$ by $V(k,p) - T^2V(k,0)$ where $T^2V(k,0)$ is the second order Taylor polynomial of $V(k,p)$ in p computed at $p = 0$ and to use power counting. Thus the quantity $H(p^2)$ has a limit as the cutoff gets removed. Furthermore the values of B and C at order g^2 are just those we defined above since at this order the only contribution is $B_{\beta_0,\mathcal{F}}$ and $C_{\beta_0,\mathcal{F}}$ for the unique forest $\mathcal{F}$ of β_0, which contributes the one consisting of the largest sub-diagram.

Exercise 18.1.2 Just the same, with $\varphi(x,\varepsilon)$ and a bound using now

$$\sup\{|\partial^k\varphi(x,\varepsilon)/\partial x^k|\ ;\ |\varepsilon|\le 1\ ;\ k\le s\ ,\ |x|\le 1\}\ .$$

Exercise 18.1.3 Indeed $\int_{-1}^{1}\mathrm{d}x/(x^2-\mathrm{i}\varepsilon)=\int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}}\mathrm{d}t/(\sqrt{\varepsilon}(t^2-\mathrm{i}))$.

Exercise 18.1.4 It is the convergence near zero which is a problem, that is the existence of

$$\lim_{\varepsilon\to 0^+}\int_0^1 \mathrm{d}r\, r^3\frac{1}{(r^2-\mathrm{i}\varepsilon)^s}=\lim_{\varepsilon\to 0^+}\varepsilon^{2-s}\int_0^{1/\sqrt{\varepsilon}}\mathrm{d}t\, t^3\frac{1}{(t^2-\mathrm{i})^s}\ ,$$

where we have set $r=\sqrt{\varepsilon}t$. This quantity just happens to have a limit when $s>2$. It is quite obvious that the limit does not exist for $s>2$ since the integral converges and the exponent of ε is negative. For $s=2$ the integral diverges. For $s<2$ the integrand grow as t^{3-2s}, so that the integral grows as $(1/\sqrt{\varepsilon})^{4-2s}=\varepsilon^{s-2}$ and the limit is finite (as one can verify by actually proving that the limit does not change if one replaces the integral by $\int_1^{1/\sqrt{\varepsilon}}\mathrm{d}t\, t^{3-2s}$).

Exercise 18.1.5 Just integrate by parts s' times in y and s times in x.

Exercise 18.1.6 After change of variable this looks like

$$\int_{\{\|y\|\le A\}}\frac{\mathrm{d}^5 y}{(\|y\|^2-\mathrm{i}\varepsilon)^2}=\sqrt{\varepsilon}\int_{\|y\|\le A\varepsilon^{-1/2}}\frac{\mathrm{d}^5 y}{(\|y\|^2-\mathrm{i})^2}\ .$$

Looking at Exercise 18.1.4 this could have a limit, but the result might depend on what one assumes about θ.

Exercise 18.2.4 Thinking of p and ε as parameters, it follows from Lemma 15.1.15 that for any space E one has $\deg_E P_M(k)\le\deg_E P(k,p,\varepsilon)$. Denoting by Q the denominator in (18.14), the necessary condition $\deg_E P(k,p,\varepsilon)-\deg_E Q<0$ for convergence implies $\deg_E P_M(k)-\deg_E Q<0$.

Exercise 18.2.9 In fact no computation is needed. As a function of x the quantity

$$G(x):=F(x,y,u)-\sum_{i\le s}u_i\Big(\sum_{j\le n}a_{i,j}\big(x_j-B_j(y,u)\big)\Big)^2$$

is a first degree polynomial in x because the quadratic terms eliminates. It is a constant because $\partial G/\partial x_i=0$ since this is the case for both terms on the right at $x=B(y,u)$. The desired result follows from computing the value at the point $x=B(y,u)$.

Exercise 18.2.10 Consider the linear map T from $\mathbb{R}^n$ to $\mathbb{R}^s$ given by $T(x)=(\sum_{j\le n}a_{i,j}x_j)_{i\le s}$ and E the image of T. The point $T(B(y,u))$ is the point $z(y,u)=(z(y,u)_i)_{i\le s}$ of E for which the function $\sum_{i\le s}u_i(z_i-y_i)^2$ of $(z_i)_{i\le s}\in E$ is minimum.

We will prove that for any matrix $(a_{i,j})_{i\le s,j\le n}$ the point $z(y,u)$ stays bounded over all values of u and of y_i with $|y_i|\le 1$. When the matrix $(a_{i,j})_{i\le s,j\le n}$ is of rank n, the map T is one-to-one, so that the point $z(y,u)$ determines the point $B(y,u)$, and these points also stay bounded over all values of u and of $|y_i|\le 1$.

The idea of the proof is simple. The point $z(y,u)$ is the point where a certain ellipsoid $\sum_{i\le s} u_i(z_i-y_i)^2\le\alpha$ of $\mathbb{R}^s$ centered at y touches the linear space E. For $z(y,u)$ to be far the ellipsoid would have to be stretched diagonally, which is not the case for our ellipsoids, which are stretched only along the axes.

The formal proof goes by induction over s. The result is obvious for $s=1$. Let us argue by contradiction, and consider a sequence $u_\ell\in\mathcal{D}'$, points $y_\ell=(y_{i,\ell})_{i\le s}$ with $|y_{i,\ell}|\le 1$, and points $z_\ell\in E$ which minimize the quantity $\sum_{i\le s} u_{i,\ell}(z_{i,\ell}-y_{i,\ell})^2$ over all the possible choices of $z_\ell\in E$. We may assume by taking a subsequence that the limits $u_i=\lim_{\ell\to\infty}u_{i,\ell}$, $y_i=\lim_{\ell\to\infty}y_{i,\ell}$ and $z_i=\lim_{\ell\to\infty}z_{i,\ell}$ exist, with the possibility $z_i=\pm\infty$. Let

$$I=\{i\le s\ ;\ u_i=0\}\ .$$

We want to rule out the possibility that one of the z_i is infinite. We observe that

$$\sum_{i\le s} u_{i,\ell}(z_{i,\ell}-y_{i,\ell})^2\le\sum_{i\le s}u_{i,\ell}y_{i,\ell}^2\le 1\ ,$$

because the choice of z_ℓ minimizes the left-hand side, and that the right-hand side is simply the value of the left-hand side for $z_\ell=0$. Taking the limit we have $\sum_{i\le s}u_i(z_i-y_i)^2\le 1$. Thus for $i\notin I$ we have z_i finite, and there is nothing to prove if I is empty. Observe also that since $\sum_{i\le s}u_{i,\ell}=1$ it holds that $\sum_{i\le s}u_i=1$ so that $\operatorname{card}I<s$. Thus the complement I^c of I is not empty. Assuming that I is not empty we consider the projection Q from $\mathbb{R}^s$ to $\mathbb{R}^{I^c}$ which forgets the coordinates x_i for $i\in I$. We can decompose $E=E_1\oplus E_2$ where $E_1\subset\ker Q$ whereas Q is one-to-one on E_2. We can therefore decompose $z_\ell=z_\ell^1+z_\ell^2$ where $z_\ell^1\in E_1$ and $z_\ell^2\in E_2$, so that $Q(z_\ell)=Q(z_\ell^2)=(z_{i,\ell})_{i\in I^c}$. Since $z_i=\lim_{\ell\to\infty}z_{i,\ell}$ is finite for $i\notin I$, as $\ell\to\infty$ the limit of $Q(z_\ell^2)=Q(z_\ell)$ exists. Thus this is also the case of the limit of $z_\ell^2=Q^{-1}(Q(z_\ell^2))$, and the numbers $z_{i,\ell}^2$ stay bounded as $\ell\to\infty$.

Now, since z_ℓ minimizes the quantity $\sum_{i\le s}u_{i,\ell}(z_i-y_{i,\ell})^2$ over all choices of $z\in E$, z_ℓ^1 minimizes the quantity $\sum_{i\le s}u_{i,\ell}(z_i^1+z_{i,\ell}^2-y_{i,\ell})^2$ over all choices of $z^1\in E_1$. Since $z_\ell^1\in\ker Q$ we have $z_{i,\ell}^1=0$ for $i\notin I$. This implies that z_ℓ^1 minimizes the quantity

$$\sum_{i\in I}u_{i,\ell}(z_i^1+z_{i,\ell}^2-y_{i,\ell})^2$$

among all the possible points of $z^1\in E_1$. That is, z_ℓ^1 is obtained by the same procedure as z_ℓ but in fewer dimensions (since $\operatorname{card}I<s$). The induction hypothesis shows that since the numbers $-z_{i,\ell}^2+y_{i,\ell}$ stay bounded, this is also the case for the numbers $z_{i,\ell}^1$. Hence the numbers $z_{i,\ell}$ stay bounded as desired.

Exercise 18.2.11 A function of x given by a formula of the type (18.29) does attain its minimum on $\mathbb{R}^n$, although this minimum may not be reached at a unique point.

Let us fix y. For each x we have $G(y,u_n) \le F(x,y,u_n)$, so that as $u_n \to u$ we get

$$\limsup_{n\to\infty} G(y,u_n) \le F(x,y,u)$$

and therefore $\limsup_{n\to\infty} G(y,u_n) \le G(y,u)$.

Next if $u_n \in \mathcal{D}$, it holds that for a certain x_n we have $G(y,u_n) = F(x_n,y,u_n)$. Taking a subsequence we may assume that $x = \lim x_n$ exists, so that $\liminf_{n\to\infty} G(y,u_n) \ge F(x,y,u) \ge G(y,u)$.

Thus we have proved that whenever $u_n \in \mathcal{D}'$ is a converging sequence, $\lim_{n\to\infty} G(y,u_n)$ exists. It is then an exercise in elementary topology to show that the function on $\mathcal{D}$ defined by $U(u) = \lim_{n\to\infty} G(y.u_n)$ whenever $u_n \to u$ is continuous on $\mathcal{D}$, i.e. the function $u \mapsto G(u,y)$ defined on $\mathcal{D}'$ extends to a continuous function on $\mathcal{D}$. We have proved that for each y the function $u \mapsto G(y,u) = \sum_{j,j'\le n} F_{j,j'}(u) y_j y'_j$ extends to a continuous function on $\mathcal{D}$. It then follows from the interpolation principle of Lemma 18.2.13 that this is also the case of the functions $F_{j,j'}(u)$.

Exercise 18.2.17 In this situation an elementary computation is the most effective. Let us assume that $Q(y) = \sum_{i,j\le r} c_{i,j} y_i y_j$ so that

$$-Q(p^0) + \sum_{1\le\nu\le 3} Q(p^\nu) = \sum_{i,j\le r} \eta_{\mu,\nu} c_{i,j} p_i^\mu p_j^\nu \ ,$$

where as usual repeated Lorentz indices are summed. Replacing each p_i by $L(p_i)$ for a Lorentz transformation L replaces the right-hand side by

$$\sum_{i,j\le r} \eta_{\mu,\nu} c_{i,j} L^\mu{}_\lambda p_i^\lambda L^\nu{}_{\lambda'} p_j^{\lambda'} = \sum_{i,j\le r} \eta_{\lambda,\lambda'} c_{i,j} p_i^\lambda p_j^{\lambda'} \ ,$$

since $\eta_{\nu,\mu} L^\nu{}_\lambda L^\mu{}_{\lambda'} = \eta_{\lambda,\lambda'}$.

Exercise 18.3.1 Use exactly the previous estimates on (18.28). In particular the denominator is $\ge (\|k\|^2+1)^s/C$.

Exercise 18.3.2 Since $B(q,u) = (B_j(q,u))_{j\le n}$ it suffices to prove that $A(B_j(q,u)) = B_j(A(q),u)$. Denoting by $a^\nu{}_\mu$ the matrix of A, we have $A(B_j(q,u)) = (a^\nu{}_\mu B_j(q^\mu,u))_{0\le\nu\le3} = (B_j(a^\nu{}_\mu q^\mu),u)_{0\le\nu\le3} = (B_j(A(q)^\nu,u)_{0\le\nu\le3} = B_j(A(q),u)$ where we have used that the map $q \mapsto B_j(q,u)$ is linear.

Exercise 19.3.10 Consider an element β of $\bar{\mathcal{F}}$ which meets α. Since $\bar{\mathcal{F}}$ is a forest either β contains α (in which case we are done) or α strictly contains β, so that α contains β^+. If α strictly contains β^+ we are done by Lemma 19.3.5 (a) because β^+ is contained in one of the α_i. If $\beta^+ = \alpha$, then, by construction β is one of the components τ_j of the diagram τ of the basic construction. But since all the edges of

$\mathcal{E}(\mathcal{F},\alpha)$ are α-active, the diagram τ is jsut the union of the α_i so that its connected components are the maximal sub-diagram α_i themselves and β is one of them.

Exercise A.2.2 First by Lemma A.2.1 this projective representation arises from a projective representation which is strongly continuous in a neighborhood of zero. Then by Lemma A.1.1 it arises from a representation which is a true representation in a neighborhood of zero, and the result follows easily.

Exercise A.5.10 (a) First, given a character w, for any $b \in N$ we have $\sum_{a\in N} w(a) = \sum_{a\in N} w(ab) = w(b)\sum_{a\in N} w(a)$. Thus if there exists b with $w(b) \neq 1$ then $\sum_{a\in N} w(a) = 0$. If w' is another character, then $a \to w'(a)^* w(a)$ is a character which takes a value different from 1 so that $\sum_{a\in N} w'(a)^* w(a) = 0$. So if we have a linear combination $\sum_w \alpha_w w \equiv 0$, i.e. $\sum_w \alpha_w w(a) = 0$ for each a then for another character w' we have $0 = \sum_w \sum_a \alpha_w w'(a)^* w(a) = \alpha_{w'}$. (b) Each character is a function $N \to \mathbb{C}$ and the dimension of this space of functions is $\operatorname{card} N$. Since the characters are linearly independent we have $\operatorname{card} \widehat{N} \leq \operatorname{card} N$. But each element a of N defines a character on $\widehat{N}$ by the map $w \to w(a)$ and this defines an injection from N to $\widehat{M}$ where $M = \widehat{N}$. Thus $\operatorname{card} N \leq \operatorname{card} \widehat{M} \leq \operatorname{card} M = \operatorname{card} \widehat{N}$.(c) Just repeat the arguments of Section 9.8. (d) We could repeat the arguments of the proof of Proposition 9.4.6, but the proof is even simpler. Consider $f \in \mathcal{F}$ with $f \neq 0$. Assuming that $(\lambda(a,A)f, g) = 0$ for each a, A we have to show that $g = 0$. Thus for each (a, A) we assume that $\sum_{B\in H} \widehat{\kappa}(B)(w^\sharp)(a)(f(AB), g(B)) = 0$. For $w \in O$ let $S_w = \{B \in H; \widehat{\kappa}(B)(w^\sharp) = w\}$, so that $\sum_B \widehat{\kappa}(B)(w^\sharp)(a)(f(AB), g(B)) = \sum_{w\in O} w(a) \sum_{B\in S_w} (f(AB), g(B))$. As this holds for each a we then conclude from (a) that $\sum_{B\in S_w} (f(AB), g(B)) = 0$ for each w and each A. Now if $B, B' \in S_w$ we have $C := B^{-1}B' \in H_{w^\sharp}$, so that $B' = BC$ and thus $f(AB') = U(C^{-1})f(AB)$ and similarly $g(BB') = U(C^{-1})f(B)$. Since U is unitary this shows that $(f(AB), g(B))$ is independent of $B \in S_w$. So $(f(AB), g(B)) = 0$ for each $A \in H$ and each $B \in S_w$. Since w is arbitray, $(f(A), g(B)) = 0$ for each A, B in H. Now since $f \neq 0$ there exists D with $f(D) \neq 0$, and since U is irreducible the set of $f(DC) = U(C^{-1})(f(D))$ for $C \in H_{w^\sharp}$ spans $\mathcal{V}$. In particular the set of $f(A)$ spans $\mathcal{V}$ so that $g(B) = 0$ for each B.

Exercise A.5.11 For $A, B \in H$ we have $\widehat{\kappa}(A)(w^\sharp) = \widehat{\kappa}(B)(w^\sharp)$ if and only if $\widehat{\kappa}(B^{-1}A)(w^\sharp) = w^\sharp$ i.e. if and only if $B^{-1}A \in H_{w^\sharp}$. Thus the map $A \to \widehat{\kappa}(A)(w^\sharp)$ from H to O factors through $H/H_{w^\sharp}$, providing a natural identification of O and $H/H_{w^\sharp}$. The rest should be obvious.

Exercise A.5.12 (a) This is simply because the restriction of Ξ to $\mathcal{V}$ and $H_{w^\sharp}$ is U and Ξ has the same dimension as $\mathrm{Ind}_{H_{w^\sharp}}^{H}(U)$, see Theorem A.4.1. The argument for (b) is identical.

Exercise A.6.3 The relation (A.47) should be pretty obvious in a basis where $b \in SU(2)$ is diagonal.

Exercise C.1.5 All statements are straightforward to check, including (c) if one reads (C.10) as $T(s,t,r)T(s',t',r') = T((s,t,r)*(s',t',r'))$.

Exercise C.2.3 From (C.22) one gets

$$\langle\gamma|\gamma\rangle = \exp\Big(-\frac{|\gamma|^2}{\hbar^2}\Big)\sum_{k,n\geq 0}\frac{1}{n!k!}\frac{|\gamma|^2}{\hbar^2}\langle 0|a^k(a^\dagger)^n|0\rangle\ ,$$

and it is a special case of Wick's theorem (Lemma 13.8.1) that $\langle 0|a^k(a^\dagger)^n|0\rangle = \delta_k^n k!$. (In one word, a term on the right-hand side of (13.41) can be non-zero only if it pairs $a^\dagger$ and a so one must have $k=n$ and there are $k!$ way to make such pairs.)

Exercise C.2.4 To look for the elements x such that $ax = \gamma x/\hbar$ try to find x of the type $\sum_{n\geq 0} c_n(a^\dagger)^n|0\rangle$, using that $a(a^\dagger)^n = (a^\dagger)^n a + n(a^\dagger)^{n-1}$ to find that $c_n = \gamma c_{n-1}/(n\hbar)$.

Exercise C.2.6 The equality

$$(-1)^n\exp(y^2)\frac{\mathrm{d}^n}{\mathrm{d}y^n}\exp(-y^2) = \exp(y^2/2)\Big(y-\frac{\mathrm{d}}{\mathrm{d}y}\Big)^n\exp(-y^2/2)$$

is proved by induction over n. Differentiating both sides in y and rearranging the result yields the same equality for $n+1$ rather than n. To prove that $H_n(y) = (-1)^n\exp(y^2)\frac{\mathrm{d}^n}{\mathrm{d}y^n}\exp(-y^2)$ it suffices from (C.26) to prove that

$$\exp(2uy-u^2) = \exp(y^2)\sum_{n\geq 0}(-1)^n\frac{u^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\exp(-y^2)$$

or, equivalently,

$$\exp(-(y-u)^2) = \sum_{n\geq 0}\frac{(-u)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\exp(-y^2)\ ,$$

which is simply Taylor's formula used for the function $v\mapsto\exp(-(y+v)^2)$ at $v=0$.

Exercise C.2.7 Use the relations $\sqrt{n!}e_n = (a^\dagger)^n|0\rangle$ so that by definition of P_n $e_n(x) = P_n(\alpha x)\varphi_0(x)/\sqrt{n!}$ and $P_n(y) = H_n(\sqrt{2}y)/2^{n/2}$ by definition of H_n.

Exercise C.2.8 As we have seen, for $\gamma = u + \mathrm{i}v$ we have $A(\gamma) = S(-2u\beta, 2v\alpha)$ so $|\gamma\rangle = S(2u\beta,-2v\alpha)^{-1}|0\rangle$ and combining (C.13) and (C.16) yields $\Delta_\gamma^2 X\Delta_\gamma^2 P = \Delta_0^2 X\Delta_0^2 P (= \hbar^2/4)$ is as small as permitted by Heisenberg's uncertainty principle (2.12). Furthermore by (C.11) and (C.14) we get $\langle\gamma\exp(\omega t)|X|\gamma\exp(\omega t)\rangle = 2\beta\mathrm{Re}\,(\gamma\exp(\omega t))$ and $\langle\gamma\exp(\omega t)|P|\gamma\exp(\omega t)\rangle = -2\alpha\mathrm{Im}\,(\gamma\exp(\omega t))$. These functions are of the type $c\cos\omega(t-t_0)$ and $-cm\omega\sin\omega(t-t_0)$ just as in the case of a classical harmonic oscillator.

Exercise C.3.2 Consider $f\in L^2$, $f\neq 0$. Consider $g\in L^2$ and assume that for each s,t the integral of $\bar{g}S(s,t)f$ is zero. Setting $s=0$ and making suitable averages over t the integral of $\bar{g}f\xi$ is zero for each test function ξ, so that $\bar{g}f$ is zero a.e. and g

is zero a.e. on the support of f. Considering then translations of f, g must be zero a.e.

Exercise C.3.7 The case $P' = P + u1$ and $X' = X + v1$ is solved by (C.11) and (C.14). The case $\alpha = \delta = 0$ and $\beta = 1, \delta = -1$ is solved by the Fourier transform, according to (2.23) and (2.24). The case $P' = \rho P, X' = X/\rho$ is solved by the transformation $W(f)(x) = f(x\rho)/\rho$. The transformation $W(f)(x) = f(x)\exp(\mathrm{i}\beta x^2/2)$ takes care of the case $P' = P - \beta X$ and $X' = X$.

Exercise C.4.2 Indeed,

$$(g, P(f)) = \int \mathrm{d}x g(x)^*(-\mathrm{i}f'(x) + \mathrm{i}xf(x))\exp(-x^2)$$

and by integration by parts

$$\int \mathrm{d}x g(x)^* f'(x)\exp(-x^2) = \int \mathrm{d}x(-g'(x)^* + 2xg(x)^*)f(x)\exp(-x^2)$$

and thus

$$(g, P(f)) = \int \mathrm{d}x(-\mathrm{i}g'(x) + \mathrm{i}g(x))^* f(x)\exp(-x^2) = (P(g), f)\ .$$

Exercise C.5.1 Changing X_k into $\exp sX_k$ and P_k into $\exp(-s)P_k$ changes the operators a_k and $a_k^\dagger$ of (C.72) into the operators a'_k and $a'^\dagger_k$ of (C.76).

Exercise C.5.2 The relation (C.82) is a special case of (C.23). Since $A_k(\gamma)$ is a unitary operator, taking the adjoint of (C.82) we obtain $A_k(\gamma)a_k^\dagger A_k(\gamma)^{-1} = a_k^\dagger - \gamma^* 1$. That is, the unitary operator $A_k(\gamma_k)$ witnesses that the pairs $(a_k, a_k^\dagger)$ and $(a_k - \gamma_k 1, a_k^\dagger - \gamma_k^* 1)$ are unitarily equivalent. Thus we have (letting you guess how we define $S_k(s,t)$!) $A_k(\gamma_k)S_0(s,t)A_k(\gamma_k)^{-1} = S_k(s,t)$, i.e $S_0(s,t) = A_k(\gamma_k)^\dagger S_k(s,t)A_k(\gamma_k)$ and (C.63) is satisfied by the function $\tau_k = A_k(\gamma_k)|0\rangle$. Since $|0\rangle$ is the constant function equal to 1, we have $\int \tau_k \mathrm{d}\mu_1 = \langle 0|A_k(\gamma_k)|0\rangle$, and this equals $\exp(-|\gamma_k|^2/2)$ is a consequence of (C.22), in the form $\langle 0|\gamma\rangle = \exp(-|\gamma|^2/2)$. The rest is obvious.

Exercise C.5.3 (a) Consider a sequence (φ_k) of test functions on $\mathbb{R}^3$, with $\|\varphi_k\| = 1$, and assume that the support of φ_k is contained in $C_k = [-k-1, k+1]^3 \setminus [-k,k]^3$. Then the sequence $a_k = A(\varphi_k)$ satisfies the canonical commutation relations. As a consequence of (C.84) the sequence (a_k) is unitary equivalent to the sequence $(a_k + \gamma_k 1)$ where $\gamma_k = \int \gamma(\boldsymbol{p})\varphi_k(\boldsymbol{p})\mathrm{d}^3\boldsymbol{p}/(2\pi)^3$. According to the previous exercise, the quantity $\sum_k |\gamma_k|^2$ is finite. The function ψ_k such that $\psi_k(\boldsymbol{p}) := \gamma(\boldsymbol{p})^* \mathbf{1}_{C_k}$ satisfies $\int \gamma(\boldsymbol{p})\psi_k(\boldsymbol{p})\mathrm{d}^3\boldsymbol{p}/(2\pi)^3 = \int_{C_k} |\gamma(\boldsymbol{p})|^2 \mathrm{d}^3\boldsymbol{p}/(2\pi)^3 = \|\psi_k\|^2$. A test function φ_k with support in C_k, with $\|\varphi_k\| = 1$ and approximating $\psi_k/\|\psi_k\|$ well will satisfy $2|\gamma_k|^2 \geq \int_{C_k} |\gamma(\boldsymbol{p})|^2 \mathrm{d}^3\boldsymbol{p}/(2\pi)^3$. Since $\sum_k |\gamma_k|^2 < \infty$ this shows that $\gamma \in \mathcal{H}$.

For (b), let us choose a basis of $\mathcal{H}$ such that the linear form ψ is (a multiple of) the coordinate on the first basis vector. Then we are simply in the situation of (C.77) for $\gamma_k = 0$ if $k \geq 2$.

Exercise D.1.2 Indeed, for any integer n we then have $\exp n\varepsilon_k X \in G$. If $\varepsilon_k \to 0$ we can find a sequence n_k with $n_k\varepsilon_k \to t$ so that $\exp tX = \lim_{n\to\infty} \exp(n_k\varepsilon_k) \in G$ since G is closed. Thus $X \in \mathfrak{g}$.

Exercise D.1.10 Just write down the components of $[\boldsymbol{u}\cdot\mathbf{J}, \boldsymbol{v}\cdot\mathbf{J}]$.

Exercise D.1.11 If the unit vector $\boldsymbol{u}$ is fixed by X, then X induces an orthogonal transformation in the plane perpendicular to $\boldsymbol{u}$, so that X since $\det X = 1$ X must be a rotation. The elements of the type $\exp\theta\boldsymbol{u}\cdot\mathbf{J}$ provide all the rotations of axis $\boldsymbol{u}$.

Exercise D.1.12 Very similar to the proof of Lemma D.9.1. Namely, we have $\mathrm{d}A(t)/\mathrm{d}t = \boldsymbol{u}\cdot\mathbf{J}A(t)$ and $\mathrm{d}A^{-1}(t)/\mathrm{d}t = -A(t)^{-1}\boldsymbol{u}\cdot\mathbf{J}$ so that this derivative is $A^{-1}(t)BA(t)$ where $B = -[\boldsymbol{u}\cdot\mathbf{J}, A(t)(\boldsymbol{v})\cdot\mathbf{J}] + (\boldsymbol{u}\cdot\mathbf{J}A(t)(\boldsymbol{v}))\cdot\mathbf{J}$. Now, by (D.13) we have $[\boldsymbol{u}\cdot\mathbf{J}, A(t)(\boldsymbol{v})\cdot\mathbf{J}] = (\boldsymbol{u}\wedge A(t)(\boldsymbol{v}))\cdot\mathbf{J}$, and $\boldsymbol{u}\wedge A(t)(\boldsymbol{v}) = \boldsymbol{u}\cdot\mathbf{J}A(t)(\boldsymbol{v})$.

Exercise D.6.7 We write

$$\begin{aligned} N(V(h)(x))^2 &= \int \|V(g)V(h)(x)\|^2 \mathrm{d}\mu(g) = \int \|V(gh)(x)\|^2 \mathrm{d}\mu(g) \\ &= \int \|V(g)(x)\|^2 \mathrm{d}\mu(g) = N(x)^2 \;. \end{aligned}$$

Denote by $(\cdot,\cdot)$ the inner product in $\mathcal{H}$. Then $\langle x, y\rangle := \int\langle(V(g)(x), V(g)(y))\mathrm{d}\mu(g)$ is an inner product, and $N(x)^2 = \langle x, x\rangle$. When $\mathcal{H}$ is finite-dimensional, N and $\|\cdot\|$ are equivalent because any two norms are equivalent.

Exercise D.6.11 If σ is any of the Pauli matrices,

$$\pi_j(\exp(-\mathrm{i}t\sigma/2))(f)(z_1, z_2) = f(z_1(t), z_2(t)) \tag{P.35}$$

where $\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \exp(\mathrm{i}t\sigma/2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, so that taking the derivative of (P.35) at $t = 0$ we get $\pi'_j(-\mathrm{i}\sigma/2)(f)(z_1, z_2) = \dot{z}_1\partial f/\partial z_1 + \dot{z}_2\partial f/\partial z_2$, where $\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = (\mathrm{i}\sigma/2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then $Z = 2\mathrm{i}\pi'_j(-\mathrm{i}\sigma_3/2)$ is given by $Z(f)(z_1, z_2) = -z_1\partial f/\partial z_1 + z_2\partial f/\partial z_2$, and z_2^j is an eigenvector of eigenvalue j. Similarly, $A^+(f)(z_1, z_2) = -z_2\partial f/\partial z_1$ and $A^-(f)(z_1, z_2) = -z_1\partial f/\partial z_2$.

Exercise D.6.12 A non-zero invariant subspace has to be invariant by the operators A^+ and A^- and as in Proposition D.6.1 we show that it has to be the whole space.

Exercise D.7.5 Probably it is best to figure this out yourself with little pictures. Recall (D.55), and the definition of A_j. Obviously $m_j = 0$ for $j > n + n'$. For $j = n + n'$ then $j \in A_{n+n'}$ but to no other set A_ℓ. Next $n + n' - 1$ does not belong to any set A_ℓ, $n + n' - 2$ belongs to $A_{n+n'}$ and $A_{n+n'-2}$, etc. and in this way one gets the first equality in (D.56). To prove the second equality, we compute the last term of (D.56) and we show that it equal $1+\min(n, n', r)$. Since $k' = r - k$ we have to count the number of integers k such that $0 \le k \le \min(n, r)$ and $r - k \le n'$ i.e. $r - n' \le k$. If $n' \ge r$ this is obviously $1+\min(n, r) = 1+\min(n, n', r)$. Assume then $n' < r$. Since $\ell = n + n' - 2r \ge 0$ we have $r \le (n+n')/2$ and thus $n' < (n+n')/2$ so that $n' < r < n$. Then $\min(n, n', r) = n'$ and $\min(n, r) = r$. The number of values of k with $r - n' \le k \le r = \min(n, r)$ is then $n' + 1$.

Exercise D.7.8 Follow the hint.

Exercise D.7.9 It follows from Proposition D.7.7 (b) that the spaces $\mathcal{G}_n$ are orthogonal, so they form an orthogonal decomposition of $\mathcal{H}$. Furthermore, again for the same reason, an irreducible $\mathcal{G}$ of dimension n is orthogonal to each of the $\mathcal{G}_{n'}$ for $n' \ne n$ so it is a subspace of $\mathcal{G}_n$. Thus $\mathcal{G}_n$ is just the span of the irreducibles of dimension n.

Exercise D.9.3 By definition of κ for $X \in \mathsf{sl}_{\mathbb{C}}(2)$ one has $M(\kappa(\exp tX)(x)) = \exp tXM(x)\exp tX^\dagger$ and taking the derivative at $t = 0$ yields $M(\kappa'(X)(x)) = XM(x) + M(x)X^\dagger$, from which the desired relations are checked by explicit computation.

Exercise D.9.4 This is obvious if $v_1 = v_2 = 0$ because then $\exp \boldsymbol{v} \cdot \mathbf{Y}$ is diagonal with positive coefficients. Since $A(\exp \boldsymbol{v} \cdot \mathbf{Y})A^{-1} = \exp(\boldsymbol{v} \cdot A\mathbf{Y}A^{-1})$ one may reduce to the previous case by Lemma D.9.1.

Exercise D.10.2 Consider such an invariant subspace $\mathcal{G}$ and assume that it contains a non-zero vector $\sum_{k,\ell} \alpha_{k,\ell} e_{m,\ell}$. Consider the largest integer ℓ_0 such that not all the α_{k,ℓ_0} are zero. By successive applications of (D.89) we may assume that $\ell_0 = 0$. Consider then the smallest value k_0 of k for which $\alpha_{k_0,0} \ne 0$. Successive applications of (D.86) reduce to the case where $k = m$. Thus $e_{m,0} \in \mathcal{G}$, and then each $e_{k,\ell} \in \mathcal{G}$ by successive applications of (D.87) and (D.88).

Exercise D.10.3 This is far simpler than what it sounds. Starting with the representation $\pi = \pi_{n,m}$, we know how to define the required operators, starting with $L_j = \pi'(X_j)$, etc. and they automatically satisfy the required commutations relations. If we can find a vector e such that $Z(e) = me, W(e) = ne, A^-(e) = C^-(e) = 0$ then the construction of Proposition D.10.1 carries through to construct the whole required structure. It is quite straightforward to check that the tensor $e = (x_{i_1,\dots,i_m,j_1,\dots,j_n})$ such that $x_{i_1,\dots,i_m,j_1,\dots,j_n} = 0$ unless all the indices are equal to 1 when then $x_{i_1,\dots,i_m,j_1,\dots,j_n} = 1$, has the previous properties. All the rest is obvious.

Exercise D.10.4 It should be clear at this stage that the representation is of type $(m, 0)$ if and only if $\mathbf{B} = 0$ and of type $(0, n)$ if and only if $\mathbf{A} = 0$.

Exercise D.12.3 Recall the matrix J of Lemma 8.1.4. Then $J^{-1}C^*J = C^{\dagger-1}$ so that $A^\dagger = J^{-1}A^{*-1}J$. Thus

$$\theta(A)(MJ) = A(MJ)A^\dagger = AMA^{*-1}J\ .$$

If $W : \mathcal{M} \to \mathcal{M}$ is given by $W(M) = MJ$ this means that

$$\theta(A)W(M) = W(AMA^{*-1}) .$$

Consequently θ is equivalent to the representation r given by $r(A) : M \mapsto AMA^{*-1}$.

Exercise D.12.4 Let us consider the map $x \mapsto M(x)$ for $\mathbb{C}^4$ to $\mathcal{M}$ defined by the formula (8.19). Then by definition of κ we have $M(\kappa(A)(x)) = \theta(A)(M(x))$ i.e. $M\kappa(A) = \theta(A)M$ so that indeed κ is equivalent to θ. The rest is obvious.

Exercise D.12.9 This inverse image is a space of dimension nine which is invariant under the representation U, so indeed what else could it be? More precisely, the following holds. Assume that a finite-dimensional space decomposes as a direct sum of invariant subspaces $F_1, \ldots, F_k$, such that the restrictions of the representation to these subspaces are not equivalent. Then if a subspace F is invariant, and such that the restriction to this subspace of the representation is irreducible, it is one of the spaces of the decomposition. To see this consider the projection P_k of F onto F_k. It commutes with the representation, and its kernel is an invariant subspace of F, with therefore must be zero or the whole of F. When it is zero, the restriction of the representation to F and F_k are equivalent, so this happens for exactly one k and then $F = F_k$.

Exercise E.2.1 Let us shorten notation by writing $(\kappa\mu\nu) = \partial_\kappa F_{\mu\nu}$. Since $\partial_\kappa = \partial/\partial x^\kappa$ and since $L^{-1}(x)^\gamma = (L^{-1})^\gamma{}_\kappa x^\kappa = L_\kappa{}^\gamma x^\kappa$, by the chain rule the tensor field $(\kappa\mu\nu)(x)$ is transformed into the tensor field $L_\kappa{}^\gamma L_\mu{}^\lambda L_\nu{}^\alpha(\gamma\lambda\alpha)(L^{-1}(x))$. Consequently the field $(\kappa\mu\nu) + (\mu\nu\kappa) + (\nu\mu\kappa)$ is transformed into

$$L_\kappa{}^\gamma L_\mu{}^\lambda L_\nu{}^\alpha\big((\gamma\lambda\alpha) + (\lambda\alpha\gamma) + (\alpha\lambda\gamma)\big) .$$

Exercise E.2.3 $\cos\theta e_1 + \sin\theta e_2 \pm \mathrm{i}(-\sin\theta e_1 + \cos\theta e_2) = \exp(\mp\mathrm{i}\theta)(e_1 \pm \mathrm{i}e_2)$

Exercise G.1.2 Just use that $\cosh s + \sinh s + 1 = 1 + \exp s = 2\exp(s/2)\cosh(s/2)$ and $1 + \cosh s = 2\cosh(s/2)^2$.

Exercise G.1.4 Just compute $D_p D_{Pp}$ using (G.4) and the formulas $M(p) + M(Pp) = 2p_0 I$, $M(p)M(Pp) = m^2c^2 I$.

Exercise L.2.2 Use that

$$|f(x)h(x)| \le \frac{1}{(1+|x|)^2}(1+|x|)^{k+2}|f(x)| \le \frac{1}{(1+|x|)^2}\|f\|_{k+2}$$

and integrate in x.

Exercise L.2.5 Let $\mathcal{F}$ be the class of closed sets which are support of a given tempered distribution Φ, and F the intersection of this family. It is closed under finite intersection by Lemma L.2.4. If a function $\xi \in \mathcal{S}^n$ has a compact support which does not intersect F, then its support does not intersect one of the elements of $\mathcal{F}$, so $\Phi(\xi) = 0$. Thus F supports Φ.

Exercise L.2.7 Since $\delta'(\xi) = -\delta(\xi') = -\xi'(0)$ this quantity need not be zero even if ξ is 0 on the support $\{0\}$ of δ i.e. if $\xi(0) = 0$.

Exercise L.2.9 One simply shows that a non-zero distribution cannot be zero on each test function with compact support, an obvious consequence of Lemma L.2.8.

Exercise M.4.7 Since $\check{\xi}^\sharp(x) = \int \exp \mathrm{i}(p,x)\xi(p)\mathrm{d}^4x$ we have formally

$$\int \xi(p)\hat{W}(p)\mathrm{d}^4p = \hat{W}(\xi) = W(\check{\xi}^\sharp) = \int \xi(p)\Big(\int \exp \mathrm{i}(p,x)W(x)\mathrm{d}^4x\Big)\mathrm{d}^4p\ .$$

Exercise M.4.14 We write $1 = \int_0^\infty \exp((1-m)x)\mathrm{d}\rho(m)$. Given $m_0 < 1$ the right-hand side is $\geq \exp((1-m_0)x)\int_0^{m_0}\mathrm{d}\rho(m)$ and letting $x \to \infty$ this shows that $0 = \int_0^{m_0}\mathrm{d}\rho(m)$ so that ρ is constant on the interval $[0, m_0]$ and then on the interval $[0,1[$. Thus we have $1 = \int_1^\infty \exp((1-m)x)\mathrm{d}\rho(m)$. Letting $x \to \infty$ shows that ρ has a jump of 1 at $m = 1$ (i.e. that $\mathrm{d}\rho$ gives mass 1 to the point 1). We then get the identity $0 = \int_{]1,\infty[}\exp((1-m)x)\mathrm{d}\rho(m)$, and this obviously implies that ρ is constant on the interval $]1,\infty[$.

Exercise N.2.2 (a) Using formulas such as $\widehat{\partial_x f} = \mathrm{i}p_1\widehat{f}(\boldsymbol{p})$ one obtains $\widehat{\nabla f}(\boldsymbol{p}) = -\boldsymbol{p}^2\hat{f}(\boldsymbol{p})$ so that if f satisfies the Laplace equation it holds $\boldsymbol{p}^2\hat{f}(\boldsymbol{p}) = -1$. The rest is straightforward.

Exercise O.1 The first part of (O.3) shows that to get a non-zero contribution each term $a(p)$ has to be paired with a term $\varphi_a^\dagger(x)$, but in S_1 there is only one such term which cannot be paired to both $a(p_1)$ and $a(p_2)$.

Exercise O.2 We now have two copies of φ_a, and to get a non-zero result, each of them has to be paired with either an incoming a-particle or an outgoing $\bar{a}$ particle.

Exercise O.3 The value of the first diagram is

$$(-\mathrm{i})^3 g^2 (2\pi)^4 \frac{\delta^{(4)}(p_4 + p_3 - p_1 - p_2)}{m_c^2 - (p_1 - p_3)^2}\ ,$$

and for the second diagram one has to replace $p_1 - p_3$ by $p_1 - p_4$.

Exercise O.4 Bis repetita placent.