Are Many Small Sets Explicitly Small?

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ABSTRACT

We discuss various aspects of a conjecture that spans Analysis, Probability and Combinatorics. We find it interesting enough to offer a $1000 prize for a solution of any of the main problems.

Categories and Subject Descriptors

G.3 [**Mathematics of Computing**]: Probability and Statistics; G.2 [**Mathematics of Computing**]: Discrete Mathematics; G.m [**Mathematics of Computing**]: Miscellaneous

General Terms

Theory

1. WHAT THIS IS ALL ABOUT

A central issue in mathematics and computer science is to gain understanding of subsets of spaces with many dimensions. Our natural intuition fails miserably at this. A fundamental fact is:

$$\text{If a subset } A \text{ of a space with many dimensions is "large", then "most" of the points of the space are "close to" } A. \quad (1.1)$$

This is the underlying philosophy of the theory of concentration of measure, one of the truly great ideas of Analysis and Probability. In (1.1), "large" and "most" will mean with respect to some probability measure. There is much more room on how to define "close to", and some new tentative definitions will be presented below. A very nice introduction to the fundamental theory of concentration of measure can be found in [3]. The most useful results are "dimension-free", in the sense that the provide inequalities that do not involve the dimension of the underlying space (which is the ultimate goal for not having to worry about large dimension). This is in particular the case for the Gaussian concentration of measure (see [3]) or the "convex-distance inequality" of [7] (see also [3]).

If the conjectures we propose turn out to be true, we will apparently learn something fundamentally new about the complexity of sets of large measure in the product of many spaces. On the other hand, to disprove these conjecture would possibly require genuinely new examples of "large sets". It therefore seems to be worthwhile to either prove or disprove them. For the reader in a hurry to cash a check, the main conjectures of the paper are Conjecture 2.1 and its "discrete version", Conjecture 7.1. The most extravagant and transformational result would be a positive solution of Problem 10.3. Our goal is to explain these conjectures with the proper background, and then to attempt to find their "proper formulation". Along the way, we will use the unique opportunity presented by this paper to mention three other of our favorite problems in Probability, all with a strong combinatorial flavor.

2. THE CONVEXITY CONJECTURE

Consider a subset A of $\mathbb{R}^M$. How many operations are required to build the convex hull of A from A? Of course the exact answer should depend on what exactly we call "operation", but Caratheodory's theorem asserts that any point in the convex hull of A is in the convex hull of a subset B of A such that $\mathrm{card} B = M + 1$ (and this cannot be improved) so we should expect that the number of operations is of order M, and that this cannot really be improved.

Suppose now that, in some sense, A is "large", and that, rather than wanting to construct all the convex hull of A, we only try to construct "a proportion of it". Do we really need a number of operations that grows with M? First, how do we tell that a set is "large"?

Probably the most natural measure on $\mathbb{R}^M$ is the canonical Gaussian measure $\gamma = \gamma_M$, i.e. the law of a sequence $(g_i)_{i \le M}$ of i.i.d. standard Gaussian r.v.s.

To simplify the discussion, let us assume that the set A is balanced, (or, if one prefers, is symmetric and star-shaped) i.e.

$$x \in A \ , \ |\lambda| \le 1 \Rightarrow \lambda x \in A. \quad (2.1)$$

Conjecture 2.1. *There exists $\varepsilon > 0$ and an integer q with the following property. For any M, and any compact balanced set $A \subset \mathbb{R}^M$ with $\gamma(A) \ge 1 - \varepsilon$, we can find a convex compact set*

$$B \subset \underbrace{A + \cdots + A}_{q \text{ times}} \quad (2.2)$$

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with $\gamma(B) \geq 1/2$, where of course the right hand side is the set of sums $\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_q$ for $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_q$ in A.

The word "compact" and the choice of the constant $1/2$ are completely unimportant.

For large M the measure γ is basically concentrated on the sphere of radius $\sqrt{N}$, and this can be used to (easily) show that one obtains an equivalent conjecture if one replaces in Conjecture 2.1 the Gaussian measure γ by the uniform measure on the unit sphere. The same comment applies to Conjecture 3.3 below.

If this conjecture were true this would mean that q operations suffice to build a large convex set from any large set, *irrespective* of the dimension of the underlying space.

Wouldn't this be fundamental? The conjecture is discussed at length in the paper [5], but has attracted no attention. The present work is a renewed attempt to popularize this and related questions.

What support do we have for Conjecture 2.1? The truth is very little. The author tried to disprove it. He proved in [5] that $q = 2$ does not work, even if we replace $A + A$ by $L(A + A)$ for an arbitrarily large constant L. Whether $q = 3$ works is apparently open. We encourage the reader to spend a little time on this question. It is rather instructive to write a few natural examples of sets A for which one knows that $\gamma(A) > 9/10$ and to look at $A + A + A$. It turns out that this set is really huge in all the examples we could think of. Of course, the true obstacle is there: we have very little intuition of what really is a set of large measure in a high-dimensional space. A positive or negative solution of Conjecture 2.1 certainly qualifies for the \$1000 prize. It seems that if the conjecture is wrong, there is a chance that a counter-example to it would be a rather new kind of set and would be quite instructive. The same comments apply to the main conjectures in the rest of the paper.

3. GAUSSIAN PROCESSES

Certain early results of the author on Gaussian processes motivate the formulation of some of our conjectures, and the present section reviews these results. The reader who is not interested in either Gaussian or empirical processes can jump directly to Section 7, where the basic conjectures 7.1 and 7.2 are formulated in terms of very simple objects.

From the abstract point of view we use, a (centered) Gaussian process is a family $(X_t)_{t \in T}$ of jointly Gaussian r.v.s and the quantity of fundamental importance is $\mathsf{E} \sup_{t \in T} X_t$. It is discussed in detail in the first pages of [8], which contains a rather complete account of the author's ideas on stochastic processes, and let us simply say here that this is the natural measure of the "size" of the process. This processes induces a canonical distance d on the index set T, given by

$$d(s,t)^2 = \mathsf{E}(X_s - X_t)^2.$$

It turns out that, within a multiplicative factor, the quantity $\mathsf{E} \sup_{t \in T} X_t$ can be computed as a "functional of the geometry of the metric space (T, d)", i.e.

$$\frac{1}{L}\gamma_2(T,d) \leq \mathsf{E} \sup_{t \in T} X_t \leq L\gamma_2(T,d) \tag{3.1}$$

where L is a number and the quantity $\gamma_2(T, d)$ depends only on the geometry of the metric space (T, d). (Its exact definition is not important here and is explained at length in [8].) Thus (3.1) transforms a probabilistic problem (evaluating $\mathsf{E} \sup_{t \in T} X_t$) into a geometric problem (evaluating $\gamma_2(T, d)$). It is important to fully understand the nature of this result: it performs no magic. It is very difficult to estimate $\gamma_2(T, d)$ when the combinatorics of the space (T, d) are complicated. The meaning of (3.1) is simply that there is **no other way** to estimate $\mathsf{E} \sup_{t \in T} X_t$ than to estimate $\gamma_2(T, d)$.

At this point we make a half-column digression to mention that there remains a nice unsolved problem concerning the computation of $\gamma_2(T, d)$ in a rather explicit and concrete case. The author has proposed the following "the Ultimate Matching Conjecture" ([8], p. 110) that would (do a lot more than) nicely interpolate between the Ajtai-Komlòs-Tusnàdy matching theorem and the Leighton-Shor grid matching theorem.

CONJECTURE 3.1. *There exists a number L with the following property. Consider $\alpha_1 > 0$ and $\alpha_2 > 0$ with $1/\alpha_1 + 1/\alpha_2 = 1/2$. Then, given two independent i.i.d. samples $(X_i^1, X_i^2)_{i \leq N}$ and $(Y_i^1, Y_i^2)_{i \leq N}$ uniform in the unit square, with probability that goes to 1 as $N \to \infty$ there exists a one to one map π on $1, \ldots, N$ such that for $j = 1, 2$ we have*

$$\sum_{1 \leq i \leq N} \exp\left(\sqrt{\frac{N}{\log N}} \frac{|X_i^j - Y_{\pi(i)}^j|^{\alpha_j}}{L}\right) \leq 2.$$

The case $\alpha_1 = \alpha_2 = 4$ is of special interest. If it where true, with probability close to 1 we would find a matching π that satisfies, for some universal constant L^* both

$$\max_{i,j} |X_i^j - Y_{\pi(i)}^j| \leq L^* \frac{(\log N)^{3/4}}{N^{1/2}}$$

as in the Leighton-Shor grid matching theorem and

$$\sum_{i \leq N} |X_i^j - Y_{\pi(i)}^j| \leq L^* (\log N)^{1/2} N^{1/2}$$

as in the Ajtai-Komlòs-Tusnàdy matching theorem. Conjecture 3.1 (at least in certain crucial cases) essentially boils down to controlling $\gamma_2(T, d)$ where T is a certain class of functions on the unit square (which is defined through a control of the partial derivatives) and where d is the natural distance induced by $L_2([0, 1]^2)$, see [8].

After this parenthesis, we go back to Gaussian processes. An essentially equivalent way to look at these is, given M, to consider the canonical process $(X_{\boldsymbol{t}})_{\boldsymbol{t} \in \mathbb{R}^M}$ where the basic probability space is $(\mathbb{R}^M, \gamma)$ (recalling that γ is the canonical Gaussian measure) and where for $\boldsymbol{x} \in \mathbb{R}^M$ we have

$$X_{\boldsymbol{t}}(\boldsymbol{x}) = \boldsymbol{t} \cdot \boldsymbol{x}, \tag{3.2}$$

the dot product of $\boldsymbol{t}$ and $\boldsymbol{x}$. In that case the canonical distance induced by the process is the Euclidean distance on $\mathbb{R}^M$.

Consider a convex balanced subset T of $\mathbb{R}^M$. Let us define

$$U = \mathsf{E} \sup_{\boldsymbol{t} \in T} X_{\boldsymbol{t}} = \int \sup_{\boldsymbol{t} \in T} X_{\boldsymbol{t}}(\boldsymbol{x}) d\gamma(\boldsymbol{x}), \tag{3.3}$$

where the process $X_{\boldsymbol{t}}$ is as in (3.2). (So, one might say that U measures the size of T with respect to the canonical Gaussian process.) Since $\sup_{\boldsymbol{t} \in T} X_{\boldsymbol{t}} \geq 0$ because T is balanced, it follows from Markov's inequality that for any number $v \geq 1$ one has

$$\gamma\Big(\Big\{\sup_{\boldsymbol{t} \in T} X_{\boldsymbol{t}} \geq vU\Big\}\Big) \leq \frac{1}{v}. \tag{3.4}$$

On the other hand, it is proved in [8] as a (non trivial) consequence of (3.1) that the following holds true.

PROPOSITION 3.2. *There exists a number L such that, (whatever the value of M and the choice of the set T) one can find a sequence $(H_k)_{k\ge 1}$ of half-spaces of $\mathbb{R}^M$ such that, with the notation* (3.3),

$$\left\{\sup_{t\in T} X_{\boldsymbol{t}} \ge LU\right\} \quad \subset \quad \bigcup_k H_k \tag{3.5}$$

$$\sum_{k\ge 1} \gamma(H_k) \quad \le \quad \frac{1}{2}. \tag{3.6}$$

Of course a half-space is simply a set of the type $\{\boldsymbol{x};\ \boldsymbol{t}\cdot\boldsymbol{x} \ge 1\}$ for some $\boldsymbol{t} \in \mathbb{R}^M$. What this means is that not only the set $\{\sup_{\boldsymbol{t}\in T} X_{\boldsymbol{t}} \ge LU\}$ in (3.5) has a small measure for γ (which we know from (3.4)), but we have a kind of *explicit witness* of this, this set is covered by a union of simple sets (half-spaces) such that the *sum* of their measures is $\le 1/2$. (Of course, the choice of the number $1/2$ is pretty arbitrary.) The information (3.5) is, in a sense, less fundamental than (3.1), but, on the other hand, it is still very much non trivial.

Only simple arguments about Gaussian measures are required to show from (3.5) that one can reformulate Conjecture 2.1 as follows.

CONJECTURE 3.3. *There exists an integer q with the following property. Given any integer M and any balanced set $A \subset \mathbb{R}^M$ with $\gamma(A) \ge 1 - 1/q$, there exists a sequence (H_k) of half-spaces of $\mathbb{R}^M$ such that*

$$\boldsymbol{x} \notin \underbrace{A + \cdots + A}_{q \text{ times}} \Rightarrow \boldsymbol{x} \in \bigcup_{k\ge 1} H_k$$

and

$$\sum_{k\ge 1} \gamma(H_k) \le \frac{1}{2}.$$

We are not saying that the proper way to approach this conjecture is to look for a magic wand that would produce the half-spaces H_k. The point of the formulation of Conjecture 3.3 is to motivate further statements.

4. EMPIRICAL PROCESSES

Having proved (3.1) one would certainly like to find similar results for other classes of stochastic processes (a question that is explored in depth in [8]). Empirical processes constitute an important such class. They are certainly related to computer science, in particular in their rôle in dimension-reduction theorems as in [4]. The basic setting is as follows. Consider a sequence of i.i.d. uniformly distributed r.v.s Y_i in $[0,1]$ and $\mathcal{F}$ a (finite) class of (Borel) functions. We assume for simplicity that $\int_0^1 f(x)dx = 0$ for each $f \in \mathcal{F}$. For $f \in \mathcal{F}$, consider the r.v.

$$X_f = \frac{1}{\sqrt{N}} \sum_{i\le N} f(Y_i). \tag{4.1}$$

The problem is to estimate the size of the r.v.

$$\sup_{\mathcal{F}} X_f = \sup_{f\in\mathcal{F}} \frac{1}{\sqrt{N}} \sum_{i\le N} f(Y_i), \tag{4.2}$$

say to bound its expectation. Given f, the central limit theorem shows that for large N the r.v. X_f is nearly Gaussian. Thus, a natural way to control the process $(X_f)_{f\in\mathcal{F}}$ is because it is close to a Gaussian process (that we understand through (3.1)). There is however a fundamentally different reason why the r.v. (4.2) can be small: it is because the larger r.v.

$$\sup_{f\in\mathcal{F}} \frac{1}{\sqrt{N}} \sum_{i\le N} |f(Y_i)| \tag{4.3}$$

is already small. (In contrast, in the near-Gaussian case, one expects that there will be a lot of cancellation between terms and that this contributes in an essential way to the smallness of the sum.)

It can be shown in a precise way that the two previous methods are the *only* two ways to control the sup of the process $(X_f)_{f\in\mathcal{F}}$ [6]. Since the near-Gaussian case is well understood, the question is to understand the case (4.3), or, equivalently (since $N^{-1/2}$ is a normalization factor), to understand the quantity

$$\mathsf{E} \sup_{f\in\mathcal{F}} \sum_{i\le N} f(Y_i), \tag{4.4}$$

where $\mathcal{F}$ consists of functions $f \ge 0$. Ideally we would like to be able to compute (4.4) as a function of the "geometry" of $\mathcal{F}$; but it is at present very unclear what would be the proper way to look at $\mathcal{F}$. Then the natural thought is that, if one cannot prove anything as satisfactory as (3.1), one should be less ambitious and try for something of the nature of (3.5). Another natural though is that, instead of (4.4), one should probably think first about a more basic problem of the same nature. We keep in mind that we are trying to prove inequalities, and that to do this there is no loss to replace continuous structures by ("suitably fine") discrete structures. Let us divide the interval $[0,1]$ into M ($\gg N$) consecutive small intervals, and let us assume that each function of $\mathcal{F}$ is constant in these small intervals. Choosing the r.v.s $(Y_i)_{i\le N}$ is pretty much the same (at least modulo a poissonization argument) as selecting each small interval with probability $p = N/M$.

5. SELECTOR PROCESSES

The previous considerations bring us naturally to *selector processes.* In a selector process we consider a finite set S and i.i.d. r.v.s $(\delta_i)_{i\in S}$ with $\mathsf{P}(\delta_i = 1) = p$ and $\mathsf{P}(\delta_i = 0) = 1 - p$. (The name arises from the fact that this process allows one to choose (= select) a random subset of size $\simeq p\mathrm{card}S$ of S). Consider a class $\mathcal{F}$ of non negative functions on S or, equivalently, a class T of sequences $\boldsymbol{t} = (t_i)_{i\in S}$ with $t_i \ge 0$. We are interested in the quantity

$$\mathsf{E}(T) := \mathsf{E} \sup_{\boldsymbol{t}\in T} \sum_{i\in S} t_i \delta_i. \tag{5.1}$$

The situation of interest is where

$$\mathsf{E} \sup_{\boldsymbol{t}\in T} \sum_{i\in S} t_i\delta_i \gg p \sup_{\boldsymbol{t}\in T} \sum_{i\le N} t_i = \sup_{\boldsymbol{t}\in T} \mathsf{E} \sum_{i\in S} t_i\delta_i. \tag{5.2}$$

Before we pursue the discussion, let us explain the notation we will use throughout the paper. Let us denote by $\mu = \mu_p = \mu_{p,S}$ the law of the sequence $(\delta_i)_{i\in S}$. It is a product measure on $\{0,1\}^S$. It is convenient to think of points in

$\{0,1\}^S$ as subsets of S. We will denote by I, J, X, Y subsets of S, while A and B will denote subsets of $\{0,1\}^S$. Thus we have

$$\mathsf{E}(T) = \mathsf{E}\sup_{\boldsymbol{t}\in T}\sum_{i\in S} t_i\delta_i = \mathsf{E}_p \sup_{\boldsymbol{t}\in T}\sum_{i\in X} t_i.$$

The notation in the right-hand side means that we take expectation when $X \subset S$ is seen as a random element of the probability space $(\{0,1\}^S, \mu_p)$, and this is the notation we will use throughout the rest of the paper.

The problem of evaluating the quantity (5.2) has been partially solved in the very special case where T consists of sequences $\boldsymbol{t}$ with $t_i \in \{0,1\}$. This is explained in [9], a paper quite related to the present work. To introduce notation appropriate to that case, for a subset A of $\{0,1\}^S$ we define

$$\mathsf{E}_p(A) = \mathsf{E}_p \sup_{J\in A} \operatorname{card}(J\cap X), \tag{5.3}$$

which is simply the quantity $\mathsf{E}(T)$ of (5.1) for $T = \{\mathbf{1}_J\ ; J \in A\}$.

We will now make a one page digression to introduce the reader to some of the author's favorite obsessions about stochastic processes. We hope that the reader will find these questions beautiful and challenging. Our failure to solve them is a direct motivation for the less ambitious conjectures that are described in the present paper.

A first method to control the quantity (5.3) is through the obvious bound

$$\mathsf{E}_p(A) \le \max_{J\in A}\operatorname{card} J. \tag{5.4}$$

A second method is as follows.

Lemma 5.1. *Assume that for a certain number $M \ge 1$ we have*

$$\sum_{J\in A}\left(\frac{p\operatorname{card} J}{M}\right)^M \le 1. \tag{5.5}$$

Then $\mathsf{E}_p(A) \le LM$, where L is a universal constant.

Proof. We use the "union bound" to write for any integer k

$$\mathsf{P}\Big(\sup_{J\in A}\operatorname{card}(J\cap X) \ge k\Big) \le \sum_{J\in A}\mathsf{P}(\operatorname{card}(J\cap X)\ge k).$$

Now

$$\mathsf{P}(\operatorname{card}(J\cap X)\ge k) \le \binom{\operatorname{card} J}{k}p^k \le \left(\frac{Cp\operatorname{card} J}{k}\right)^k,$$

using the elementary estimate

$$\left(\frac{m}{Ck}\right)^k \le \binom{m}{k} \le \left(\frac{Cm}{k}\right)^k, \tag{5.6}$$

where C is a universal constant. Now, by (5.5) we have $p\operatorname{card} J \le M$ for $J \in A$, so that for $k \ge 2CM$ we get, using simply that $k \ge M$ in the last inequality,

$$\left(\frac{Cp\operatorname{card} J}{k}\right)^k \le \left(\frac{p\operatorname{card} J}{2M}\right)^k \le 2^{-k}\left(\frac{p\operatorname{card} J}{M}\right)^M,$$

and combining these estimates with (5.5) that

$$\mathsf{P}\Big(\sup_{J\in A}\operatorname{card}(J\cap X)\ge k\Big) \le 2^{-k},$$

from which the result follows easily, using that since $M \ge 1$, the smallest integer $k \ge 2CM$ satisfies $k \le 3CM$. $\square$

Having found two genuinely different methods to bound $\mathsf{E}_p(A)$ we can use *mixtures* of them. This method is based on the observation that

$$\operatorname{card}(X\cap(I_1\cup I_2)) \le \operatorname{card}(X\cap I_1) + \operatorname{card}(X\cap I_2).$$

Therefore, if we consider $A_1, A_2 \subset \{0,1\}^S$ such that

$$\forall I \in A\ ,\ \exists I_1\in A_1\ ,\ I_2\in A_2\ ,\ I\subset I_1\cup I_2, \tag{5.7}$$

we have $\mathsf{E}_p(A) \le \mathsf{E}_p(A_1) + \mathsf{E}_p(A_2)$. We can then apply the bound (5.3) to A_1 and the bound of Lemma 5.1 to A_2. More formally, let us define $\Psi(A)$ as the infimum of the numbers $M \ge 1$ for which we can find classes A_1 and A_2 such that (5.7) holds as well as

$$\max_{J\in A_1}\operatorname{card} J \le M\ \ ;\ \ \sum_{J\in A_2}\left(\frac{p\operatorname{card} J}{M}\right)^M \le 1.$$

We have just shown that

$$\mathsf{E}_p(A) \le 2\Psi(A). \tag{5.8}$$

The natural question is whether the previous method is in full generality the best possible to bound $\mathsf{E}_p(A)$, i.e. whether the previous inequality can be reversed. Since $M \ge 1$ this can be true only when $\mathsf{E}_p(A) \ge 1$. (The case where $\mathsf{E}_p(A) \le 1$, such as when A consists of one single set that contains one single point is uninteresting.)

Conjecture 5.2. *There exists a universal constant L such that if $\mathsf{E}_p(A) \ge 1$ then $\Psi(A) \le L\mathsf{E}_p(A)$.*

The following statement is a more explicit form of the same conjecture.

Conjecture 5.3. *There exists a universal constant L with the following property. Consider any set S and any $p > 0$. Then for any $A \subset \{0,1\}^S$ with $\mathsf{E}_p(A) \ge 1$ we can find $B \subset \{0,1\}^S$ with the following properties, where $M = L\mathsf{E}_p(A)$:*

$$\forall I\in A\ ,\ \exists J\in B,\ \operatorname{card}(I\setminus J)\le M. \tag{5.9}$$

$$\sum_{J\in B}\left(\frac{p\operatorname{card} J}{M}\right)^M \le 1. \tag{5.10}$$

The difficulty in proving Conjecture 5.3 is that we are given A and we have to find B, which needs not be a subset of A. Moreover, nobody painted red the elements of B to help us find them.

In the same spirit as Conjecture 5.3 (and motivating it) is the problem of understanding the supremum of Bernoulli processes, that is the quantities $b(T) = \mathsf{E}\sup_{\boldsymbol{t}\in T}|\sum_{i\in S}\varepsilon_i t_i|$, where now the ε_i are i.i.d random signs and T is a set of sequences on S. We do not assume here that $t_i \ge 0$. (The absolute values around the sum do not make the problem fundamentally different from the similar problem without these absolute values that we considered in the Gaussian case.) There are two immediate bounds for this quantity. The first one is through the trivial relation $b(T) \le \sup_{\boldsymbol{t}\in T}|t_i|$, and the second is through a (elementary) comparison theorem with the Gaussian case, the inequality

$$b(T) \le \sqrt{2/\pi}\,\mathsf{E}\sup_{\boldsymbol{t}\in T}\Big|\sum_{i\in S} g_i t_i\Big|,$$

where now the r.v.s g_i are independent standard Gaussian (so that in principle we understand the right-hand side using (3.1)). The author calls the problem of determining whether these two methods, and their mixtures as in the case of Conjecture 5.3, are really the only way to bound $b(T)$ the **Bernoulli problem**. The formal statement is as follows.

CONJECTURE 5.4. (**The Bernoulli Conjecture**) *There exists a universal constant L such that given any finite set S and any set T of sequences indexed by S, we can find two sets of sequences T_1 and T_2 which satisfy the following three properties:*

$$\forall \boldsymbol{t} \in T_1 \ , \ \sum_{i\in S} |t_i| \le Lb(T)$$

$$\mathsf{E} \sup_{\boldsymbol{t}\in T_2} \Big| \sum_{i\in S} g_i t_i \Big| \le Lb(T)$$

$$T \subset T_1 + T_2 = \{\boldsymbol{t}_1 + \boldsymbol{t}_2 \ , \ \boldsymbol{t}_1 \in T_1, \boldsymbol{t}_2 \in T_2\}. \tag{5.11}$$

The author has devoted a significant part of his life to the study of this question and offers a \$5000 prize for its solution. Amazingly enough, the program of constructing a decomposition such as in (5.11) can be carried out in many situations, as is detailed in [8].

We go back to the main story. It would be very interesting to be able to estimate in complete generality the quantity (5.1) as a function of the "geometry of T", by proving something in the spirit of Conjectures 5.3 and 5.4. This seems hopeless unless one first gives a positive solution to these conjectures, so it is apparently a very difficult project. Therefore, we should try first to look for weaker results. For this we will be guided by Proposition 3.2.

In order to distinguish conveniently between subsets of S and subsets of $\{0,1\}^S$, we will call a subset of $\{0,1\}^S$ **a class**.

We will say that a class $A \subset \{0,1\}^S$ is an *up-class* if

$$X \in A, Y \supset X \Rightarrow Y \in A.$$

We note that if A is an up-class, the map $p \mapsto \mu_p(A)$ is non-decreasing. We will say that a class $A \subset \{0,1\}^S$ is a *down-class* if

$$X \in A, Y \subset X \Rightarrow Y \in A.$$

If A is an up-class, its complement is a down-class, and conversely.

Given a subset I of S we consider the class

$$H_I = \{J \subset S; \ I \subset J\} \tag{5.12}$$

so that H_I is an up-class and

$$\mu_p(H_I) = p^{\operatorname{card} I}. \tag{5.13}$$

The notation H_I is motivated by the fact that these classes somewhat correspond to the half-spaces of Section 3. They are canonical, which motivates the following definition.

DEFINITION 5.5. *A class $A \subset \{0,1\}^S$ is p-small if there exists a family $\mathcal{I}$ of subsets of S such that*

$$A \subset \bigcup_{I\in\mathcal{I}} H_I \tag{5.14}$$

and

$$\sum_{I\in\mathcal{I}} \mu_p(H_I) = \sum_{I\in\mathcal{I}} p^{\operatorname{card} I} \le \frac{1}{2}. \tag{5.15}$$

Of course, the choice of the value $1/2$ is rather arbitrary. The idea of this definition is (again) that the sets H_I provide *explicit simple witnesses* that $\mu_p(A) \le 1/2$. It is particularly well adapted to the case where A is an up-class. Many natural classes are up-classes, such as in (5.17) below, and there is little hope to understand the structure of general classes unless we first understand that of up-classes.

Let us point out the following obvious fact.

LEMMA 5.6. *If A is p-small and $p' < p$ then A is p'-small.*

CONJECTURE 5.7. *There exists a number L with the following property. Consider any $0 < p < 1$, any set S and any set T of sequences $(t_i)_{i\in S}$ with $t_i \ge 0$. Let*

$$\mathsf{E}_p(T) = \mathsf{E}_p \sup_{\boldsymbol{t}\in T} \sum_{i\in X} t_i := \int \sup_{\boldsymbol{t}\in T} \sum_{i\in X} t_i d\mu_p(X). \tag{5.16}$$

Then the class

$$\Big\{X; \ \sup_{\boldsymbol{t}\in T} \sum_{i\in X} t_i \ge L\mathsf{E}_p(T)\Big\} \tag{5.17}$$

is p-small.

A positive answer would give a version of Proposition 3.2 for selector processes. The practical-minded reader will ask what use would such a result have, and might not share the author's feeling that it would provide fundamental information. In some sense it is a result of the same nature than (3.1). It would show that if you are given a selector process, and would like to prove that, within a multiplicative factor that is a universal constant, the quantity $\mathsf{E}_p(T)$ (defined in (5.1)) satisfies $\mathsf{E}_p(T) \le M$ for a certain number M, there is in the end **no other way** than to find the witnesses that the set $\{X; \ \sup_{\boldsymbol{t}\in T} \sum_{i\in X} t_i \ge LM\}$ is small.

One must realize that in Conjecture 5.7, the problem is for $p \le 1/2$. For $p \ge 1/2$, we leave it as a teaser the fact that the class (5.17) is empty for $L = 4$. (It could be easier to figure this out after reading Lemma 7.5 below.) Also, one feels that what matters is the case where p is sufficiently small, but we cannot quite prove it, and this will be discussed in the last section of the paper.

PROPOSITION 5.8. *With the notation of Conjecture 5.7, given a number $V > 0$, if we request that there exists a number $L(V)$ depending on V only such that the class*

$$\Big\{X; \ \sup_{\boldsymbol{t}\in T} \sum_{i\in X} t_i \ge L(V)\mathsf{E}_p(T)\Big\} \tag{5.18}$$

is (Vp)-small for any $0 < p < 1$, any set S and any set T of sequences $(t_i)_{i\in S}$ with $t_i \ge 0$, the resulting conjecture is equivalent to Conjecture 5.7.

Proof. Assuming Conjecture 5.7 and given $V > 0$, let us prove that the class (5.18) is (Vp)-small if the number $L(V)$ is large enough. Using Lemma 5.6 we can assume that $V \ge 1$. Consider an integer $n > V$. It suffices to consider the case where $2np < 1$ for otherwise, as shown in Lemma 7.5 below, the class (5.18) is empty if $L(V)$ has been chosen large enough. Then, setting $p' = 1 - (1-p)^{2n}$ one proves that

$$\mathsf{E}_{p'} \sup_{\boldsymbol{t}\in T} \sum_{i\in X} t_i \le 2n\mathsf{E}_p(T).$$

(We leave the proof inequality as a teaser for the reader who really wishes to penetrate this material. A hint can be found in Lemma 7.4 below.) It then follows from Conjecture 5.7 (used for p' rather than p) that the class (5.18) is p'-small if $L(V) = 2Ln$ where L is the constant of Conjecture 5.7. Now, since $2np < 1$ we have $(1-p)^{2n} \le \exp(-2np) \le 1 - np$ so that $p' \ge np \ge Vp$. $\square$

In Proposition 11.2 below we will show that Conjecture 5.7 holds when T consists of a collection of sequences with disjoint support. However the case where T consists of a single constant sequence is already of interest, and we give the argument right away to ensure that the reader does not miss this important computation (which was already implicitly performed in the proof of Lemma 5.1).

LEMMA 5.9. *There exists a universal constant L_1 such that for each set $J \subset S$ the class $A = \{X \subset S;\ \mathrm{card} X \cap J \ge L_1 p\, \mathrm{card} J\}$ is p-small.*

Proof. Consider the smallest integer k with $k \ge L_1 p\, \mathrm{card} J$, and

$$\mathcal{I} = \{I \subset J;\ \mathrm{card} I = k\}$$

so that, if $m = \mathrm{card} J$, using (5.6)

$$\sum_{I \in \mathcal{I}} p^{\mathrm{card} I} = p^k \binom{m}{k} \le \left(\frac{Cmp}{k}\right)^k. \tag{5.19}$$

Thus, if $L_1 \ge 2C$, then $k \ge 2Cpm$ and the quantity (5.19) is $\le 2^k \le 1/2$. On the other hand,

$$\mathrm{card}(X \cap J) \ge L_1 pm \Rightarrow \mathrm{card}(X \cap J) \ge k \Rightarrow \exists I \in \mathcal{I}, I \subset X,$$

so $\mathcal{I}$ witnesses that A is p-small. $\square$

6. WEAKLY SMALL CLASSES

One problem when studying p-small classes is that one really wonders what kind of magic wand should be used to produce the family $\mathcal{I}$. The purpose of the present section is to introduce a related notion with which it might be easier to work. For two subsets I, Y of S, let us write

$$\begin{cases} \psi(I,Y) = 1 & \text{if } I \subset Y \text{ (i.e. } Y \in H_I), \\ \psi(I,Y) = 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

DEFINITION 6.1. *We say that a class $A \subset \{0,1\}^S$ is weakly p-small if there exists a probability measure θ on $\{0,1\}^S$ such that*

$$A \subset \{\Psi \ge 1\} \tag{6.2}$$

where

$$\Psi(Y) = \frac{1}{2} \int p^{-\mathrm{card} I} \psi(I,Y) d\theta(I). \tag{6.3}$$

Of course, a probability measure on $\{0,1\}^S$ is simply a system of weights β_I for $I \subset S$, with $\beta_I \ge 0$ and $\sum_{I \subset S} \beta_I = 1$.

Since $\int \psi(I,Y) d\mu_p(Y) = \mu_p(H_I) = p^{\mathrm{card} I}$ Fubini Theorem implies that $\int \Psi(Y) d\mu_p(Y) = 1/2$. Thus when A is weakly p-small we have a concrete witness that A is small: the "somewhat simple" function Ψ is ≥ 1 on A, and of integral $\le 1/2$. This function is somewhat simple because it is a convex combination of the simple functions $\psi(I, \cdot)$.

PROPOSITION 6.2. *If a class is p-small, it is weakly p-small.*

Proof. Consider a class $A \subset \{0,1\}^S$ and a family $\mathcal{I}$ of subsets of S that satisfies (5.14) and (5.15). Consider a probability measure θ on $\{0,1\}^S$ such that $\theta(\{I\}) \ge 2p^{\mathrm{card} I}$ for each $I \in \mathcal{I}$. Such a probability measure exists by (5.15), and $\Psi(Y) \ge 1$ for $Y \in H_I$ as seen from (6.3) by restricting the integral to $\mathcal{I}$. $\square$

CONJECTURE 6.3. *There exist a number L such that every weakly p-small class is (p/L)-small.*

This seems to be a very nice problem of combinatorics. It sets aside the main theme of this paper (which is to try to discover some new structure of general up-classes) and in that sense it is not related to the main conjectures of the present work. It focuses on the issue of pulling the rabbit (i.e. the family $\mathcal{I}$) out of the hat. The reader is invited to analyze first the cases where θ is uniform over subsets of S with a given cardinality, or uniform over a class of disjoint subsets of S with the same cardinality to understand what seems to be going on.

The case where θ is carried by the sets of cardinality 1 is not difficult to settle, but is not completely trivial. It follows in particular from Theorem 11.1 below, but a simpler argument is as follow. Assume without loss of generality that $S = \{1, \dots, N\}$ and that the sequence $a_i = \theta(\{i\})$ is non-increasing. For $k \ge 1$ define $n_k = \lfloor k/p \rfloor$. Consider that class $\mathcal{I} = \cup_{k \ge 1} \mathcal{I}_k$, where $\mathcal{I}_k$ consists of the subsets of $\{1, \dots, n_k\}$ of cardinality k. Then if $X \subset S$ is not a subset of $\cup_{I \in \mathcal{I}} H_I$, the k-th element m_k of X is $> n_k$, and $\sum_{i \in X} a_i \le \sum_k a_{n_k+1} \le p \sum_i a_i \le p$. Consequently if $\sum_{i \in X} a_i > p$ then $X \subset \cup_{I \in \mathcal{I}} H_I$, where $\sum_{I \in \mathcal{I}} (p/L)^{\mathrm{card} I} \le 1/2$ if L is a large enough constant.

We could not yet decide the case where θ is carried by the sets of cardinality 2, and even the following subcase: Assume that S is the disjoint union of two subsets S_1 and S_2. Consider a probability measure θ^* on $S_1 \times S_2$, and the image θ of θ^* under the map $(i,j) \in S_1 \times S_2 \mapsto \{i,j\} \subset S$. To understand the depth of the problem, the reader might give a try to the case where S_1 and S_2 are disjoint unions of q sets $S_{1,1}, \dots, S_{1,q}$ and $S_{2,1}, \dots, S_{2,q}$ respectively and where θ is uniform over $\cup_{j \le q} S_{1,j} \times S_{2,j}$.

There exists also some natural situations suggested by the theory of random graphs (and the paper [2]) that are not obvious to analyze. Consider for example the case where S is the set consisting of the$N(N-1)/2$ edges of a complete graph with N elements, so that a subset X of S is a graph, and consider the case where θ is uniform over the sets of k-cliques. (A k-clique is a complete subgraph on a subset of k vertices.) It is probable that many such situations should be analyzed before one forms enough intuition to decide Conjecture 6.3 one way or the other.

CONJECTURE 6.4. *There exists a number L such that with the notations of Conjecture 5.7 the class* (5.17) *is weakly p-small.*

In view of Proposition 6.2 this is weaker than Conjecture 5.7. Should Conjecture 6.3, have a positive answer, Proposition 5.8 shows that these conjectures would be equivalent, but of course it could happen that Conjecture 5.7 is true but Conjecture 6.3 is false.

As in Proposition 5.8 one can show that one gets the same conjecture if one requests that the class (5.17) is, say, $2p$-small.

The definition of weakly small classes is motivated by the following.

PROPOSITION 6.5. *Consider a class $A \subset \{0,1\}^S$. Then the following are equivalent.*

a) The class A is not weakly p-small.

b) There exists a probability measure ν on A such that

$$\forall I \subset S, \qquad \nu(H_I) < 2p^{\mathrm{card} I}. \tag{6.4}$$

Proof. Let us assume b), so that, recalling the notation (6.1), we have

$$\int \psi(I,Y)d\nu(Y) = \nu(H_I) < 2p^{\mathrm{card} I}. \tag{6.5}$$

Considering a probability measure θ on $\{0,1\}^S$, by (6.3), (6.5) and Fubini Theorem we have

$$\int \Psi(Y)d\nu(Y) < 1,$$

so that since $\nu(A) = 1$ we have $A \not\subset \{\Psi \geq 1\}$. Thus A is not weakly p-small. Therefore b) $\Rightarrow$ a).

Conversely, let us assume that A is not weakly p-small. This implies that any convex combination of functions of the type $(1/2)p^{-\mathrm{card} I}\psi(I,\cdot)$ takes a value < 1 on A. The Hahn-Banach theorem then shows that A carries a probability measure ν for which the integral of any of these functions is < 1, which means that (6.4) holds. $\square$

DEFINITION 6.6. *We say that a probability measure ν on $\{0,1\}^S$ is δ-spread if it satisfies*

$$\forall I \subset S, \qquad \nu(H_I) \leq \delta^{\mathrm{card} I}. \tag{6.6}$$

In Definition 6.6 we have removed the factor 2 that occurs in (6.4). This factor is inappropriate for tensorization and is inelegant. The cost of doing this is that one cannot use Proposition 6.5 to characterize exactly weakly p-small classes by saying that they do not carry a p-spread measure. However we have the following, which is almost as nice, since the loss of a factor 2 is totally irrelevant here.

PROPOSITION 6.7. *Consider a class $A \subset \{0,1\}^S$. Then*

a) If the class A is not weakly p-small it carries a $2p$-spread probability measure.

b) If the class A carries a δ-spread probability measure it is not weakly δ-small.

Proof. We use Proposition 6.5 and we notice that $2p^{\mathrm{card} I} \leq (2p)^{\mathrm{card} I}$ when $\mathrm{card} I > 0$. $\square$

In that spirit, we will show that Conjecture 6.4 can be reformulated as follows.

CONJECTURE 6.8. *There exists a number L with the following property. Consider any number $0 < p < 1$, any set S, any class $A \subset \{0,1\}^S$. Assume that for $J \in A$ we are given a sequence $\boldsymbol{t}^J = (t_i^J)_{i \in S}$ with $t_i^J \geq 0$ and $\sum_{i \in J} t_i^J \geq 1$. Then if there exists a p-spread probability measure ν with $\nu(A) = 1$ we have (with the notation (5.16))*

$$\mathsf{E}_p \sup_{J \in A} \sum_{i \in X} t_i^J \geq \frac{1}{L}. \tag{6.7}$$

The point is that we only have $\sup_{J \in A} \mathsf{E}_p \sum_{i \in X} t_i^J \geq p$, so that (6.7) is a non-trivial statement.

The reformulation of Conjecture 5.7 as Conjecture 6.8 brings forward its true nature: it would be a new kind of lower bound on selector processes. In the last section of the paper we will prove a few results supporting Conjecture 6.8, in particular Corollary 11.4. This Corollary implies (but this is in fact already proved in [9]) that in the case of special interest when $\mathrm{card} J = n$ is a given integer for all $J \in A$, and where $t_i^J = 1/n$ for $i \in J$, then (6.7) holds. In other words,

$$\mathsf{E}_p(A) = \mathsf{E}_p \sup_{J \in A} \mathrm{card}(X \cap J) \geq n/L. \tag{6.8}$$

Proof that Conjecture 6.4 implies Conjecture 6.8. We set $T = \{\boldsymbol{t}^J \; ; \; J \in A\}$, so that for any $X \subset S$ we have

$$\sup_{J \in A} \sum_{i \in X} t_i^J = \sup_{\boldsymbol{t} \in T} \sum_{i \in X} t_i,$$

and consequently

$$\mathsf{E}_p(T) = \mathsf{E}_p \sup_{\boldsymbol{t} \in T} \sum_{i \in X} t_i = \mathsf{E}_p \sup_{J \in A} \sum_{i \in X} t_i^J.$$

We notice that since $\sum_{i \in J} t_i^J \geq 1$ we have

$$A \subset \Big\{X \; ; \; \sup_{J \in A} \sum_{i \in X} t_i^J \geq 1\Big\} = \Big\{X \; ; \; \sup_{\boldsymbol{t} \in T} \sum_{i \in X} t_i \geq 1\Big\}.$$

Since A carries a p-spread probability measure, A is not p-small by Proposition 6.7 b) and if Conjecture 6.4 holds true the class A cannot be contained in the class (5.17) i.e. $L\mathsf{E}_p(T) > 1$, which is (6.7). $\square$

Proof that Conjecture 6.8 implies Conjecture 6.4. We may assume that $p \leq 1/2$. Consider a set T of sequences $(t_i)_{i \in S}$ with $t_i \geq 0$ and, defining $\mathsf{E}_p(T)$ as in (5.16), consider a number V to be determined later and the class

$$A = \Big\{J \; ; \; \sup_{t \in T} \sum_{i \in J} t_i \geq V\mathsf{E}_{2p}(T)\Big\}.$$

By definition of A

$$\forall J \in A \; , \;\; \exists (u_i^J)_{i \in S} \in T \; , \; \sum_{i \in J} u_i^J \geq V\mathsf{E}_{2p}(T). \tag{6.9}$$

Consider $J \in A$ and define $\boldsymbol{t}^J = (t_i^J)_{i \in S}$ by $t_i^J = u_i^J$ if $i \in J$ and $t_i = 0$ otherwise. Thus for any $X \subset S$ we have

$$\sup_T \sum_{i \in X} t_i \geq \sum_{i \in X} u_i^J \geq \sum_{i \in X} t_i^J$$

and hence in particular

$$\mathsf{E}_{2p} \sup_T \sum_{i \in X} t_i \geq \mathsf{E}_{2p} \sup_{J \in A} \sum_{i \in X} t_i^J. \tag{6.10}$$

Assume for contradiction that there exists a $2p$-spread probability measure ν on A. Then, using (6.7) (for $2p$ rather than p), (6.9) and homogeneity, we get

$$\mathsf{E}_{2p} \sup_{J \in A} \sum_{i \in X} t_i^J \geq \frac{V\mathsf{E}_{2p}(T)}{L},$$

and combining with (6.10) :

$$\mathsf{E}_{2p}(T) = \mathsf{E}_{2p} \sup_T \sum_{i \in X} t_i \geq \frac{V \mathsf{E}_{2p}(T)}{L}.$$

If e.g $V = 2L$, this is impossible, so that A cannot carry a $(2p)$-spread probability measure, and by Proposition 6.7 it is p-small. $\square$

One intrinsic difficulty in this circle of questions is that it is hard to analyze δ-spread probability measures.

We first study a fundamental example. Let $M = \mathrm{card} S$, and consider the probability measure ν on $\{0,1\}^S$ that gives mass $1/M$ to each subset J of S of cardinality 1. Then ν is δ-spread for $\delta = 1/M$. To see this we note that $\nu(H_I) = 0$ unless $\mathrm{card} I = 1$, and that when $\mathrm{card} I = 1$ we have $\nu(H_I) = \delta$.

On the other hand, ν is supported by the up-class

$$A = \{X;\ \mathrm{card} X \geq 1\}$$

and $\mu_p(A) = 1 - (1-p)^M$. This is almost 1 for large M so A is not small for μ_p; but condition (6.6) is stable by tensorization; that is, if ν_1 on $\{0,1\}^{S_1}$ and ν_2 on $\{0,1\}^{S_2}$ satisfy this conditions so does $\nu_1 \otimes \nu_2$ on $\{0,1\}^{S_1} \times \{0,1\}^{S_2} = \{0,1\}^{S_1 \cup S_2}$ (here we assume that $S_1 \cap S_2 = \varnothing$). Thus, tensorization r times gives a measure ν that satisfies (6.6), but is supported by an up-class A with $\mu_p(A) = (1 - (1-p)^M)^r$. For r large, $\mu_p(A)$ is as small as one wishes while A carries a δ-spread probability measure so is not weakly δ-small.

This example seem to have been brought to light in [1] (and I learned it through Noga Alon). Let us give two generalizations of it. In the first generalization we fix $k \leq M = \mathrm{card} S$, and ν gives mass $\binom{M}{k}^{-1}$ to each subset J of S of cardinality k. Then $\nu(H_I) = 0$ if $\mathrm{card} I > k$ while if $\mathrm{card} I = n \leq k$ we have

$$\nu(H_I) = \frac{\binom{M-n}{k-n}}{\binom{M}{k}} = \frac{k(k-1)\cdots(k-n+1)}{M(M-1)\cdots(M-n+1)} \leq \left(\frac{k}{M}\right)^n.$$

In the second generalization, ν gives mass $1/N$ to each subset of a collection of N disjoint subsets of S of cardinality k. Then ν is δ-spread for $\delta = N^{-1/k}$.

To construct more complicated examples one can remark that the class of δ-spread positive measures is closed under the following two operations. Consider for $j = 1, 2$ a probability measure ν_j that lives on $\{0,1\}^{S_j}$ and is δ-spread, where S_1 and S_2 are disjoint sets. Then, as already pointed out, $\nu_1 \otimes \nu_2$ is δ-spread. Also, if Δ_j is the probability measure on $\{0,1\}^{S_j}$ that is concentrated at $\varnothing$, and if $\nu_1' = \nu_1 \otimes \Delta_2$ and $\nu_2' = \nu_2 \otimes \Delta_1$, then any probability measure $\nu \leq \nu_1' + \nu_2'$ is δ-spread. These claims are very simple to check. Iteration of these operations yields complicated examples, but we will show latter that none of these can be of help to disprove the main conjecture (Conjecture 7.8) of the next section.

Let us give a last example to demonstrate further the kind of complication that can arise in the structure of δ-spread measures. We consider three integers M, r and s. Consider a set S which is the disjoint union of r sets $S_1, \ldots, S_r$ of cardinality M, and

$$A = \{I \subset S\ ;\ \forall j \leq r, \mathrm{card}(I \cap S_j) = 1\}.$$

Thus $\mathrm{card} A = M^r$. Let us consider a set S', disjoint from S, of cardinality sM^r, and a map $\varphi : A \to \{0,1\}^{S'}$ such that the sets $\varphi(I)$ for $I \in A$ are each of cardinality s and form a partition of S'. Finally let

$$B = \{J \subset S \cup S'\ ;\ J \cap S \in A, J \cap S' = \varphi(J \cap S)\}.$$

Then it is quite straightforward to see that the uniform probability ν on B is δ-spread for $\delta = M^{-r/(r+s)}$.

7. POSITIVITY AND MORE DARING CONJECTURES

Consider a set T of sequences $\boldsymbol{t} = (t_i)_{i \in S}$ and for $X \subset S$ define

$$\varphi(X) = \sup_{\boldsymbol{t} \in T} \sum_{i \in X} t_i. \tag{7.1}$$

When $t_i \geq 0$ for every $\boldsymbol{t}$ in T, it is then clear that

$$\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y). \tag{7.2}$$

Given an integer q and a class $A \subset \{0,1\}^S$, let us define

$$A^{(q)} = \{Y \subset S;\ \forall X_1, \ldots, X_q \in A,\ Y \not\subset X_1 \cup \cdots \cup X_q\}.$$

In words $A^{(q)}$ is the class of subsets of S that cannot be covered by q sets in A. It is an up-class. It follows from (7.2) that if

$$A = \{\varphi < a\}$$

then

$$\{\varphi \geq qa\} \subset A^{(q)}.$$

Now if $a = b\mathsf{E}_p\varphi$ $(= b \int \varphi(X) d\mu_p(X))$ we have $\mu_p(A) \geq 1 - 1/b$.

One should wonder whether Conjecture 5.7 would not follow from the more general fact that $A^{(q)}$ is p-small whenever $\mu_q(A)$ is not small.

Conjecture 7.1. *There exists a number q with the following property. For each $0 < p < 1$ and each class A we have*

$$\mu_p(A) \geq 1 - \frac{1}{q} \Rightarrow A^{(q)} \text{ is } p\text{-small.} \tag{7.3}$$

Let us observe that even when q is very large, it is not true that $\mu_p(A^{(q)})$ is very small. This is shown by the trivial example $A = \{\emptyset\}$, where for each q we have $A^{(q)} = A^c$.

One has the following weaker version of conjecture 7.1.

Conjecture 7.2. *Same as above, but requiring only*

$$\mu_p(A) \geq 1 - \frac{1}{q} \Rightarrow A^{(q)} \text{ is weakly } p\text{-small.} \tag{7.4}$$

Proving or disproving these conjectures qualifies for the prize, but this is not the case for any of the subsequent riskier conjectures.

The most obvious construction to disprove Conjecture 7.2 would be to consider a class $B \subset \{0,1\}^S$ that carries a p-spread measure and to define

$$A = \left\{I \subset S\ ;\ \forall J \in B, \mathrm{card}(I \cap J) < \frac{1}{q}\mathrm{card} J\right\},$$

so that $B \cap A^{(q)} = \emptyset$. But (6.8) implies that this construction fails to contradict (7.3) because $\mu_p(A)$ is very small.

One might think of Conjecture 7.1 as a discrete version of Conjecture 2.1. The parallel between these conditions is more obvious when Conjecture 2.1 is reformulated as Conjecture 3.3. (This is of course how we invented Conjecture 7.1.)

As was the case in Section 4 we get the same conjecture if in the conclusion we require, say, A to be $(p/2)$-small (provided we are permitted to change the value of q).

The crucial phenomenon in the present circle of ideas is that when $\mu_p(A) \geq 1/2$, the class $A^{(2)}$ seems to be extremely small. At least this appears to be the case on all the examples we know. It is this phenomenon that must be analyzed.

Much of the rest of the paper is devoted to the effort of trying to find "the proper formulation" of Conjecture 7.2. This will lead us through a series of more and more daring conjectures, and it is quite fascinating that none of them seems easy to disprove. It is equally fascinating that none of the subsequent ways to look at the problem that we will explore seems to bring us any closer to a positive solution.

Given a number $0 < \alpha < 1$, and a subset J of S let us define the probability $\theta_{J,\alpha}$ by

$$\theta_{J,\alpha}(\{I\}) = \alpha^{\operatorname{card}(J\cap I)}(1-\alpha)^{\operatorname{card}(J\setminus I)}.$$

Thus $\theta_{J,\alpha}$ is a product measure. When $i \notin J$ the factor of $\theta_{J,\alpha}$ of rank i is concentrated at 0, and when $i \in J$ it gives mass α to 1 and $1-\alpha$ to 0. An alternate way to look at this object is to observe that if we define $A_J = \{I \subset J\ ;\ I \in A\}$, then $\theta_{J,\alpha}(A) = \mu_{J,\alpha}(A_J)$, where $\mu_{J,\alpha}$ is defined as μ_p but for J instead of S and α instead of p.

Conjecture 7.3. *There exists a number $0 < \alpha < 1$ with the following property. Consider $0 < p < 1$ and an up-class $A \subset \{0,1\}^S$. Then the class*

$$\{J \subset S\ ;\ \theta_{J,\alpha}(A) > \mu_p(A)\}$$

is weakly αp-small.

We did not check it, but it seems virtually certain as in Proposition 5.8 that we get the same conjecture if we replace the condition "weakly αp-small" by the stronger condition "weakly p-small ".

Conjecture 7.3 is in the spirit of (1.1). One can think of the class $\{J \subset S\ ;\ \theta_{J,\alpha}(A) \leq \mu_p(A)\}$ as a kind of neighborhood of A, and the meaning of Conjecture 7.3 is that this neighborhood comprises "most of the points."

The main idea underlying Conjecture 7.3 is that it could be easier to prove a statement that does not involve the (somewhat mysterious) operation $(X,Y) \mapsto X \cup Y$. Let us note that the smaller α, the weaker is Conjecture 7.3.

Our next goal is to prove that Conjecture 7.3 implies Conjecture 7.2. We start by some simple observations.

For a class $A \subset \{0,1\}^S$ and an integer q we denote by $A_{(q)}$ the complement of $A^{(q)}$ i.e.

$$A_{(q)} = \{Y;\ \exists X_1,\ldots,X_q \in A,\ Y \subset X_1 \cup \cdots \cup X_q\}. \quad (7.5)$$

We observe that this is a down-class. We also observe that if $A^{\sim}$ is the smallest down-class that contains A, i.e.

$$A^{\sim} = \{X \subset S\ ;\ \exists Y \in A,\ X \subset Y\}$$

then $A_{(q)} = A^{\sim}_{(q)}$. Thus, when proving any statement to the effect that $A_{(q)}$ is not small when one controls $\mu_p(A)$ from below, we can always assume that A is a down-class.

We note the formula

$$A^{(q_2)}_{(q_1)} = A^{(q_1 q_2)}. \quad (7.6)$$

Lemma 7.4. *We have*

$$\mu_\beta(A_{(q)}) \geq \mu_\alpha(A)^q \quad (7.7)$$

for $\beta = 1-(1-\alpha)^q$.

Proof. The map $\varphi : (X_1,\ldots,X_q) \mapsto X_1 \cup \cdots \cup X_q$ sends $\mu_\alpha^{\otimes q}$ to μ_β, so that since $A_{(q)} \supset \varphi(A \times \cdots \times A)$ we have

$$\begin{aligned} & \mu_\beta(A_{(q)}) \geq \mu_\beta(\varphi(A \times \cdots \times A)) \\ = \ & \mu_\alpha^{\otimes q}(\varphi^{-1}(\varphi(A \times \cdots \times A))) \\ \geq \ & \mu_\alpha^{\otimes q}(A \times \cdots \times A) = \mu_\alpha(A)^q. \end{aligned}$$

□

Lemma 7.5. *Given $0 < \alpha < 1$ there exists a number q_1 with the following property. If a down-class $A \subset \{0,1\}^S$ satisfies $\mu_\alpha(A) \geq 1 - 1/q_1$, then $A^{(q_1)} = \varnothing$.*

Proof. Consider q large enough so that $(1-\alpha)^q < 1/2$. Thus $\beta = 1-(1-\alpha)^q > 1/2$ and $\mu_\beta(A^{(q)}) \geq \mu_\alpha(A)^q$ by (7.7). Consider q_2 large enough so that $(1-1/q_2)^q \geq 1/2$. Hence if $\mu_\alpha(A) \geq 1 - 1/q_2$ we have $\mu_\beta(A^{(q)}) \geq 1/2$. Using (7.6) it therefore suffices to show that $A^{(2)} = \varnothing$ if A is a down-class and if $\mu_\beta(A) > 1/2$ for some $\beta \geq 1/2$. (One then use this for $A^{(q)}$ rather than A and one takes $q_1 = 2q_2$.) But then $\mu_{1/2}(A) \geq \mu_\beta(A) > 1/2$, and if A' is the image of A under the transformation that changes a subset X of S into its complement, then $\mu_{1/2}(A') = \mu_{1/2}(A) > 1/2$, so that $A \cap A' \neq \varnothing$. Thus there exists X and X' in A with $S = X \cup X'$ and therefore $A^{(2)} = \varnothing$. □

Proposition 7.6. *Under Conjecture 7.3 the following is true. There exists a number q_1 such that for each $A \subset \{0,1\}^S$ we have*

$$\mu_p(A) \geq 1 - \frac{1}{q_1} \Rightarrow A^{(q_1)} \text{ is weakly } \alpha p\text{-small.}$$

Proof. As previously observed we can assume that A is a down-class. We consider q_1 as in Lemma 7.5. Using this lemma for J rather than S yields

$$\theta_{J,\alpha}(A) > \mu_p(A) \Rightarrow \theta_{J,\alpha}(A) \geq 1 - \frac{1}{q_1} \Rightarrow J \in A^{(q_1)}$$

and consequently

$$A^{(q_1)} \subset \{J\ ;\ \theta_{J,\alpha}(A) < \mu_p(A)\},$$

so that $A^{(q_1)}$ is weakly αp-small by Conjecture 7.3. □

Proof that Conjecture 7.3 implies Conjecture 7.2. From Lemma 7.5 we can assume $p < \alpha/2$. Let $p' = p/\alpha < 1/2$. Consider q_2 large enough that

$$p' < 1/2 \Rightarrow p'' := 1 - (1-\alpha p')^{q_2} \geq p'.$$

Consider a down-class $A \subset \{0,1\}^S$. Using first that $p'' \geq p'$ and that $A_{(q_2)}$ is a down-class, and then Lemma 7.4 for $\alpha p' = p$ instead of α and q_2 instead of q we get

$$\mu_{p'}(A_{(q_2)}) \geq \mu_{p''}(A_{(q_2)}) \geq \mu_p(A)^{q_2}.$$

Thus, by Proposition 7.6, used for p' instead of p and $A_{(q_2)}$ instead of A, we have, since $\alpha p' = p$,

$$\begin{aligned} \mu_p(A) \geq \left(1 - \frac{1}{q_1}\right)^{1/q_2} & \Rightarrow \mu_{p'}(A_{(q_2)}) \geq 1 - \frac{1}{q_1} \\ & \Rightarrow A^{(q_1)}_{(q_2)} = A^{(q_1 q_2)} \text{ is weakly } \alpha p\text{-small.} \end{aligned}$$

Thus Conjecture 7.2 holds for q large enough that $q \geq q_1 q_2$ and $1 - 1/q \geq (1 - 1/q_2)^{q_2}$. $\square$

We turn to a more refined version of Conjecture 7.3, that is "stable by tensorization".

Definition 7.7. *Given two probability measures μ and μ' on $\{0,1\}^S$ we say that μ* dominates *μ' if for each up-class $A \subset \{0,1\}^S$ we have*

$$\mu(A) \geq \mu'(A).$$

Given a probability measure ν on $\{0,1\}^S$, we define a probability measure $W_\alpha(\nu)$ on $\{0,1\}^S$ by

$$W_\alpha(\nu)(A) = \int \theta_{J,\alpha}(A) d\nu(J). \tag{7.8}$$

Conjecture 7.8. *There exists a number $0 < \alpha < 1$ with the following property. Consider $0 < p < 1$ and a αp-spread probability measure ν on $\{0,1\}^S$ i.e. such that*

$$\nu(H_I) \leq (\alpha p)^{\mathrm{card} I} \tag{7.9}$$

for each subset I of S. Then the probability measure μ_p dominates the probability measure $W_\alpha(\nu)$, i.e for each up-class A it holds that

$$W_\alpha(\nu)(A) = \int \theta_{J,\alpha}(A) d\nu(J) \leq \mu_p(A).$$

This formulation is motivated by the fact that a statement of this type might be amenable to a proof by induction on the number of coordinates. We could not discover a suitable "induction hypothesis", but of course we checked that none of the measures ν constructed at the end of Section 4 disproves Conjecture 7.8. This follows from Theorem 11.10 below.

Proof that Conjecture 7.8 implies Conjecture 7.3. We assume that Conjecture 7.8 is true, and we consider an up-class $A \subset \{0,1\}^S$. Let us assume that the class

$$B = \{J \subset S \; ; \; \theta_{J,\alpha/2}(A) > \mu_p(A)\}$$

is not weakly $(\alpha p/2)$-small. By Proposition 6.5 we can find an αp-spread probability ν with $\nu(B) = 1$. Then for J in B we have

$$\theta_{J,\alpha}(A) \geq \theta_{J,\alpha/2}(A) > \mu_p(A).$$

Integrating this inequality in J with respect to ν shows that $W_\alpha(\nu)(A) > \mu_p(A)$. Thus μ_p does not dominate $W_\alpha(\nu)$, a contradiction that proves Conjecture 7.3 for $\alpha/2$ rather than α. $\square$

It seems potentially useful to point out a reformulation of Definition 7.7

Definition 7.9. *Given two probability measures μ and μ' on $\{0,1\}^S$ we say that μ can be* pushed downwards *to μ' if there exists a probability measure λ on $\{0,1\}^S \times \{0,1\}^S$ such that its first marginal is μ, its second marginal is μ', and λ is supported by the set*

$$\{(X,Y); \; X \supset Y\}.$$

This is related to the very important idea of "mass transportation".

Proposition 7.10. *The probability measure μ dominates the probability measure μ' if and only if the measure μ can be pushed downwards to the measure μ'.*

Proof. We prove only the easy part, the converse uses the Hahn-Banach theorem and is more delicate. Consider the probability measure λ as in Definition 7.9 and an up-class A. Then

$$\begin{aligned}\mu'(A) &= \lambda(\{(X,Y); \; Y \in A\}) \\ &= \lambda(\{(X,Y); \; Y \in A, \; X \supset Y\}) \\ &\leq \lambda(\{(X,Y); \; X \in A\}) = \mu(A)\end{aligned}$$

since $Y \in A$, $X \supset Y \Rightarrow X \in A$. $\square$

The point of mentioning this is that the probability measure λ of Definition 7.7 could be a useful object to consider.

Next we state a kind of "dual" formulation of Conjecture 7.8. The main idea is to now think of the quantity $\theta_{J,\alpha}(A)$ as a function of J. We recall the notation (6.1), and to lighten notation we write

$$h(I,J) = (\alpha p)^{-\mathrm{card} I} \psi(I,J) \tag{7.10}$$

if $I \neq \emptyset$ and $h(\emptyset, J) = 1$.

Conjecture 7.11. *There exists a number $0 < \alpha < 1$ with the following property. For any up-class A there exists a probability measure η on $\{0,1\}^S$ such that for any $J \subset S$ we have*

$$\theta_{J,\alpha}(A) \leq \mu_p(A) \int h(I,J) d\eta(I). \tag{7.11}$$

The present statement is related to Conjecture 6.4, and in some sense we have come a full circle since this no longer uses the idea of δ-spread measures.

Proof that Conjecture 7.11 implies Conjecture 7.8. If a probability measure μ is αp-spread, i.e. satisfies (7.9), then, using (7.8) in the first line and (7.11) in the second we have, for any up-class A

$$\begin{aligned}W_\alpha(\nu)(A) &= \int \theta_{J,\alpha}(A) d\nu(J) \\ &\leq \mu_p(A) \int \left(\int h(I,J) d\eta(I) \right) d\nu(J) \\ &= \mu_p(A) \int \left(\int h(I,J) d\nu(J) \right) d\eta(I) \\ &\leq \mu_p(A)\end{aligned}$$

since $\int h(I,J) d\nu(J) \leq 1$ by (7.9) and η is a probability. Thus μ_p dominates $W_\alpha(\nu)$, and this completes the proof. $\square$

Proof that Conjecture 7.8 implies Conjecture 7.11. Consider an up-class A, so that by Conjecture 7.8, for each αp-spread probability measure ν on $\{0,1\}^S$,

$$\int \theta_{\alpha,J}(A) d\nu(J) = W_\alpha(\nu)(A) \leq \mu_p(A).$$

Let us assume for contradiction that we cannot find a probability η as in (7.11). Consider the class $\mathcal{C}$ of functions g on $\{0,1\}^S$ for which there exists a probability measure η on $\{0,1\}^S$ such that

$$\forall J \subset S, \quad g(J) \leq \mu_p(A) \int h(I,J) d\eta(I).$$

Then $\mathcal{C}$ is a convex set, and since we assume that we cannot find η as in (7.11) we have $J \mapsto \theta_{\alpha,J}(A) \notin \mathcal{C}$. In fact, we can even find $\varepsilon > 0$ such that $\mathcal{C} \cap \mathcal{B}_\varepsilon = \varnothing$, where

$$\mathcal{B}_\varepsilon = \{f \; ; \; \forall J \subset S \; , \; |\theta_{\alpha,J}(A) - f(J)| \leq \varepsilon\}.$$

This follows from an obvious compactness argument. The Hahn-Banach theorem implies that there exists a linear functional ν on the space of functions on $\{0,1\}^S$ separating the convex sets $\mathcal{C}$ and $\mathcal{B}_\varepsilon$, in the sense that

$$\sup_{g\in\mathcal{C}} \nu(g) := \beta < \inf_{g\in\mathcal{B}_\varepsilon} \nu(g). \tag{7.12}$$

Since $g \in \mathcal{C}$ and $g' \le g$ imply $g' \in \mathcal{C}$, it follows that ν is positive, so it is a positive measure. Since $\mathcal{C}$ contains the constant function g given by $g(J) = \mu_p(A)$ for all $J \subset S$ (as is seen by taking $\eta(\{\varnothing\}) = 1$) we have $\beta > 0$, so we can as well by homogeneity assume that in (7.12) we have $\beta = \mu_p(A)$. Since for each $I \subset S$ the function $g(\cdot) = \mu_p(A)h(I,\cdot)$ is in $\mathcal{C}$, it follows from (7.12) that

$$\forall I \subset S,\;\; \nu(h(I,\cdot)) \le 1$$

i.e.

$$\forall I \subset S,\;\; \nu(H_I) \le (\alpha p)^{\mathrm{card} I}. \tag{7.13}$$

In particular if we take $I = \varnothing$ we see that ν has a mass at most one. Adding the appropriate mass at $\varnothing$ we turn ν into a probability measure without changing $\nu(H_I)$ for $I \ne \varnothing$, so (7.13) means that ν is (αp)-spread. Also by (7.12) we have $\beta = \mu_p(A) < \inf_{g\in\mathcal{B}_\varepsilon} \nu(g)$, and using this for the function $g : J \mapsto \theta_{\alpha,J}(A)$ yields, using (7.9) again

$$\beta = \mu_p(A) < \nu(g) = \int \theta_{\alpha,J}(A)d\nu(J) = W_\alpha(\nu)(A).$$

It follows that μ_p does not dominate $W_\alpha(\nu)$, a contradiction that finishes the proof. $\square$

When studying Conjecture 7.11 we face the same problem as in Conjecture 6.4. What is the magic method that will produce the probability measure η? Of course one cannot help feeling that Conjecture 7.11 is too strong to be possibly true. Yet, inspection of simple cases shows that possibly something deep is going on. We will explain how to construct the probability η in the case where $A = \{X \subset S;\ X \ne \varnothing\}$. This amounts to find coefficients $\beta_I \ge 0$ such that (setting $N = \mathrm{card} S$ and recalling the notation (7.10)), the following holds true:

$$\sum_{I\subset S} \beta_I \le \mu_p(A) = 1-(1-p)^N \tag{7.14}$$

and

$$\forall X \subset S,\;\; W_\alpha^*(A)(X) = 1-(1-\alpha)^{\mathrm{card} X} \le \sum_{I\subset X} \beta_I h(I,X). \tag{7.15}$$

Although the construction is a posteriori simple, it is connected with deeper ideas that will be explained in Section 9. Assuming without loss of generality that α is small enough that $1-\alpha \ge \exp(-2\alpha)$, and using the fact that by concavity for all y and all s we have

$$1-e^{-y} \le 1-e^{-s}+e^{-s}(y-s) = 1-(1+s)e^{-s}+e^{-s}y, \tag{7.16}$$

we see that for every s

$$\begin{aligned} W_\alpha^*(A)(X) &= 1-(1-\alpha)^{\mathrm{card} X} \le 1-\exp(-2\alpha \mathrm{card} X)\\ &\le 1-(1+s)e^{-s}+2\alpha e^{-s}\mathrm{card} X. \end{aligned}$$

Since $\mathrm{card} X = \alpha p\sum_{i\in S} h(\{i\},X)$ we see that (7.15) holds for $\beta_\varnothing = 1-(1+s)e^{-s}$, $\beta_{\{i\}} = 2\alpha^2 pe^{-s}$ for $i \in S$ and $\beta_I = \varnothing$ if $\mathrm{card} I \ge 2$. Thus

$$\sum_{I\subset S} \beta_I = 1-(1+s)e^{-s}+2\alpha^2 pNe^{-s},$$

and the optimal choice $s = 2\alpha^2 N$ gives

$$\sum_{I\subset S} \beta_I \le 1-\exp(-2\alpha^2 pN) \le 1-(1-p)^N$$

if $\alpha \le 1/2$.

8. THE KAHN-KALAI CONJECTURE

CONJECTURE 8.1. *[2] There exists a universal constant L such that for all $0 < p < 1$, if $A \subset \{0,1\}^S$ is an up-class that satisfies $\mu_p(A) \le 1/2$, then A is p'-small for $p' = p/(L\log \mathrm{card} S)$.*

Given the class A, the quantity of interest is the "threshold" $p = p_A$ defined by $\mu_p(A) = 1/2$. A positive solution to Conjecture 8.1 would mean that to compute p_A (within a logarithmic factor) there is no other way than to compute $p_A^* = \sup\{p\ ;\ A \text{ is } p\text{-small}\}$. This is very much in the spirit of (3.1) and Conjecture 5.7.

Conjecture 8.1 is discussed at length in [2]. Of course we have the following weaker version:

CONJECTURE 8.2. *As above, requiring only A to be weakly p'-small.*

Although this is difficult to explain, we feel that these conjectures are closely related to those of the previous sections, mainly in the sense that probably it requires a really new idea about up-classes to disprove them. Here are related questions .

CONJECTURE 8.3. *There exists a universal constant L with the following property. Consider a class $B \subset \{0,1\}^S$ and assume that for some integer m we have $\mathrm{card} I = m$ whenever $I \in B$. Assume that B carries a δ-spread measure ν and let $A = \{J \subset S, \exists I \in B, I \subset J\}$. Then if $p \ge L\delta\log(m+1)$ we have $\mu_p(A) \ge 1/2$.*

A more precise version of Conjecture 8.3 is as follows.

CONJECTURE 8.4. *There exists a universal constant L with the following property. Consider a class $B \subset \{0,1\}^S$ and assume that for some integer m we have $\mathrm{card} I = m$ whenever $I \in B$. Assume that B carries a δ-spread measure ν. Then for $p \ge \delta$ we have*

$$\mathsf{E}_p(B) = \mathsf{E}_p \sup_{I\in B} \mathrm{card}(X\cap I) \ge m(1-L\exp(-p/L\delta))\ .$$

To see that this is stronger than (8.3) one simply observe that if A is as in the statement of that conjecture, then $\mu_p(A) \ge 1/2$ whenever $\mathsf{E}(B) \ge m-1/2$.

We formulate next a common generalization to both Conjecture 8.2 and the conjectures of the previous sections. We recall the notation (7.8).

CONJECTURE 8.5. *There exists a universal constant L with the following property. Consider $0 < p < 1/2$ and $\delta > 0$. Then if ν is a δ-spread measure on $\{0,1\}^S$, and if*

$$\alpha \le 1-(1-p)^{\frac{1}{L\delta}} \tag{8.1}$$

then μ_p dominates $W_\alpha(\nu)$.

This conjecture is "dimension-independent", in the sense that the cardinality of S is irrelevant. The case $\delta = p$ recovers Conjecture 7.8.

Proof that Conjecture 8.5 implies Conjecture 8.2. Consider an up-class A and a δ-spread probability ν with $\nu(A) = 1$. Then under (8.1) μ_p dominates $W_\alpha(\nu)$, so that in particular $W_\alpha(\nu)(A) \le \mu_p(A)$. Therefore by (7.8) (and since $\nu(A) = 1$) we can find $J \in A$ with $\theta_{J,\alpha}(A) \le \mu_p(A)$. Since A is an up-class, for J in A we have $A \supset \{X\ ;\ X \supset J\}$. Thus we have, letting $M = \mathrm{card} S$,

$$\theta_{J,\alpha}(A) \ge \theta_{J,\alpha}(\{X; X \supset J\}) = \alpha^{\mathrm{card} J} \ge \alpha^M$$

and therefore

$$\mu_p(A) \ge \alpha^M. \tag{8.2}$$

Since $1 - p \le \exp(-p)$ if we set $\delta = p/2L \log M$ (where L is the constant of (8.1)) it holds that

$$(1-p)^{1/L\delta} \le \exp(-2\log M) \le \frac{1}{M^2}.$$

Therefore (8.1) is satisfied for $\alpha = 1 - 1/M^2$, so that (8.2) yields

$$\mu_p(A) \ge \left(1 - \frac{1}{M^2}\right)^M \ge \frac{1}{2}$$

for $M \ge 3$. Thus for $M \ge 3$, if $\mu_p(A) < 1/2$, we cannot find ν as above and A must be weakly δ-small. □

In Conjecture 8.5 the value of α can be very close to 1 (as in the previous proof) so this conjecture could be much riskier than the previous ones. It is of interest to test it on the case where, if $M = \mathrm{card} S$, ν gives mass $1/M$ to each set $I \subset S$ with $\mathrm{card} I = 1$, so that ν is $(1/M)$-spread, and (8.1) allows values of α as large as $1 - (1-p)^{M/L}$. To check that μ_p dominates $W_\alpha(\nu)$ one shows with a little work that the critical case is that of the class $A = \{I \subset S\ ;\ I \ne \emptyset\}$. It satisfies $\mu_p(A) = 1 - (1-p)^M$ and indeed $\mu_p(A) \ge W_\alpha(\nu)(A) = \alpha$ if $L \ge 1$.

9. MULTIPLICATIVE VERSION

Conjecture 9.1. *There exists a number $\alpha > 0$ with the following property. For each number $0 < p < 1/2$, each αp-spread measure ν and each up-class A we have (assuming $A \ne \{0,1\}^S$)*

$$-\int \log(1 - \theta_X(A))d\nu(X) \le \log(1 - \mu_p(A)). \tag{9.1}$$

Here $\theta_X(A) = \theta_{X,\alpha}(A)$ is as in Conjecture 7.3. The value of α is kept implicit to lighten notation. As we will soon see, this conjecture is stronger than Conjecture 7.8. One can similarly formulate a conjecture stronger than Conjecture 8.5 by asking that (9.1) holds under (8.1).

Passing to complements, (9.1) means that for any down-class A one should have

$$\mu_p(A) \le \prod_X \theta_X(A)^{\nu(\{X\})}, \tag{9.2}$$

a "multiplicative" inequality.

Let us start with an example. Consider a subset $I \subset S$, and let $n = \mathrm{card} I$. Let $A = \{X \subset S;\ \ I \not\subset X\}$ so $\mu_p(A) = 1 - p^n$. It is clear that $\theta_X(A) = 1$ if $I \not\subset X$, and $\theta_X(A) = 1 - \alpha^n$ if $I \subset X$. Thus (9.2) holds for this class A provided

$$1 - p^n \le (1-\alpha^n)^{\nu(\{X;\, X \supset I\})} = (1-\alpha^n)^{\nu(H_I)}.$$

This essentially requires that $\nu(H_I) \le (p/\alpha)^n = (p/\alpha)^{\mathrm{card} I}$. The meaning of Conjecture 9.1 is that this obviously necessary requirement is basically the only requirement for (9.2) to hold for general classes.

Throughout the rest of this section we denote by $\mathcal{D}(S)$ the class of measures ν on $\{0,1\}^S$ that satisfy (9.2) for every down-class $A \subset \{0,1\}^S$. (The dependence on α and p is kept implicit). We will show that the class $\mathcal{D}(S)$ enjoy remarkable stability properties, which seems to indicate that (9.1) is the "correct" conjecture.

Theorem 9.2. *a) If $\mathrm{card} S = 1$ and $\nu(\{1\}) \le p$, then $\nu \in \mathcal{D}(S)$ (provided $1 - \alpha \ge 1/e$.)*

b) More generally, if α is small enough and if $\nu(\varnothing) \ge 1 - p^{\mathrm{card} S}$, then $\nu \in \mathcal{D}(S)$.

c) If S is the disjoint union $S = S_1 \cup S_2$, and if $\nu_1 \in \mathcal{D}(S_1)$, $\nu_2 \in \mathcal{D}(S_2)$ then $\nu_1 \otimes \nu_2 \in \mathcal{D}(S)$.

d) More generally, assume that S is a disjoint union $S = S_1 \cup S_2$, consider $\nu_1 \in \mathcal{D}(S_1)$ and for $X \in S_1$ consider $\nu_X \in \mathcal{D}(S_2)$. Consider the measure ν on $\{0,1\}^{S_1 \times S_2}$ given by

$$\nu(A) = \int \nu_X(\{Y;\ \ (X,Y) \in A\})d\nu_1(X)$$

where $X \subset S_1$, $Y \subset S_2$. Then $\nu \in \mathcal{D}(S)$.

e) Assume that S is a disjoint union $S = S_1 \cup S_2$, and denote by Δ_j the probability on $\{0,1\}^{S_j}$ that is concentrated at $\varnothing$. Consider $\nu_1 \in \mathcal{D}(S_1)$ and $\nu_2 \in \mathcal{D}(S_2)$. Then any probability ν that satisfies $\nu \le \nu_1 \otimes \Delta_2 + \Delta_1 \otimes \nu_2$ belongs to $\mathcal{D}(S)$.

In Theorem 11.11 in the last section of the present paper, we will prove a more specialized result: If $k/N \le p$, the uniform probability ν on the sets $I \subset S$ with $\mathrm{card} I = k$ belongs to $\mathcal{D}(S)$.

A consequence of Theorem 9.2 is that, if there exists a αp-spread probability measure that does not satisfy (9.2), it cannot be constructed from simpler pieces using any of the operations of Theorem 9.2.

Corollary 9.3. *Conjecture 9.1 implies Conjecture 7.8.*

Proof. First, we observe that by the inequality $x \le -\log(1-x)$, under (9.1) we have

$$\int \theta_X(A)d\nu(X) \le \log(1 - \mu_p(A)). \tag{9.3}$$

Next, Theorem 9.2 implies that if ν satisfies (7.2), so do its powers $\nu^{\otimes n}$ (when we identify $(\{0,1\}^S)^n$ with $\{0,1\}^{S'}$ where $\mathrm{card} S' = n\mathrm{card} S$) so that, using (9.3) for $\nu^{\otimes n}$ and A^n,

$$\left(\int \theta_X(A)d\nu(X)\right)^n \le \log(1 - \mu_p(A)^n)$$

for all n, and thus

$$\int \theta_X(A) d\nu(X) \le \mu_p(A)$$

whenever A is an up-class. $\square$

Proof of Theorem 9.2. We find it convenient to use the formulation (9.2).

To prove a), we note that when card$S = 1$, the only non-trivial case to consider is $A = \{\varnothing\}$, in which case (9.2) reduces to

$$1 - p \le (1-\alpha)^{\nu(\{1\})},$$

which holds for $\alpha \le 1 - 1/e$ and $\nu(\{1\}) = p$.

To prove b) we observe that $\theta_X(A)$ decreases as X increases, so that it suffices to prove that $\mu_p(A) \le \theta_S(A)^a$ for $a = p^{\mathrm{card} S}$. Given $\mu_p(A)$, what is the smallest possible value of $\theta_S(A)$? The density of θ_S with respect to μ_p at I depends only on card$I$, and increases with card$I$. It should then be clear that if $A_k = \{I \subset S;\ \ \mathrm{card} I \le k\}$ then

$$\mu_p(A) \ge \mu_p(A_k) \Rightarrow \theta_S(A) \ge \theta_S(A_k)$$

and the whole matter is reduced to a struggle with the tails of the Binomial law, which is left to the interested reader.

To prove c), we consider a down-class $A \subset \{0,1\}^{S_1 \cup S_2}$. For $I \subset S_1$ we write

$$A_I = \{J \subset S_2;\ \ I \cup J \in A\} \tag{9.4}$$

and for $J \subset S_2$ we write

$$A_J = \{I \subset S_1;\ \ I \cup J \in A\}. \tag{9.5}$$

Since $\nu_1 \in \mathcal{D}(S_1)$, for each $J \subset S_2$ we have, lightening notation by writing $\nu_{1,X}$ rather than $\nu_1(\{X\})$,

$$\eta_1(A_J) \le \prod_{X \subset S_1} \theta_X(A_J)^{\nu_{1,X}},$$

where η_1 denotes the projection of μ_p on $\{0,1\}^{S_1}$. Using Fubini's theorem and Hölder's inequality we get

$$\mu_p(A) = \int \eta_1(A_J) d\eta_2(J) \le \int \prod_{X \subset S_1} \theta_X(A_J)^{\nu_{1,X}} d\eta_2(J)$$
$$\le \prod_{X \subset S_1} \left(\int \theta_X(A_J) d\eta_2(J) \right)^{\nu_{1,X}}, \tag{9.6}$$

where of course η_2 is the projection of μ_p on $\{0,1\}^{S_2}$.

Using Fubini's theorem again yields

$$\int \theta_X(A_J) d\eta_2(J) = \theta_X \otimes \eta_2(A) = \int \eta_2(A_I) d\theta_X(I). \tag{9.7}$$

Since $\nu_2 \in \mathcal{D}(S_2)$, writing $\nu_{2,X}$ rather than $\nu_2(\{X\})$ we get

$$\eta_2(A_I) \le \prod_{Y \subset S_2} \theta_Y(A_I)^{\nu_{2,Y}} \tag{9.8}$$

so that, using Hölder's inequality again,

$$\begin{aligned} \int \eta_2(A_I) d\theta_X(I) &\le \prod_{Y \subset S_2} \left(\int \theta_Y(A_I) d\theta_X(I) \right)^{\nu_{2,Y}} \\ &= \prod_{Y \subset S_2} \theta_{X \cup Y}(A)^{\nu_{2,Y}} \end{aligned} \tag{9.9}$$

and combining (9.6) to (9.9) finally yields,

$$\mu_p(A) \le \prod_{X \subset S_1, Y \subset S_2} \theta_{X \cup Y}(A)^{\nu_{1,X}\nu_{2,Y}}, \tag{9.10}$$

the required inequality.

To prove d), we simply replace (9.8) by

$$\eta_2(A_I) \le \prod_{Y \subset S_2} \theta_Y(A_I)^{\nu_{X,Y}}$$

(where $\nu_{X,Y} = \nu_X(\{Y\})$). We proceed as above, and instead of (9.10) we get

$$\mu_p(A) \le \prod_{X \subset S_1, Y \subset S_2} \theta_{X \cup Y}(A)^{\nu_{1,X}\nu_{X,Y}},$$

the required inequality.

To prove e), consider a down-class $A \subset \{0,1\}^S$ and its projections A_1 and A_2 on $\{0,1\}^{S_1}$ and $\{0,1\}^{S_2}$ respectively. Then we have, keeping the notation $\nu_{1,X}$ and $\nu_{2,Y}$,

$$\begin{aligned} \mu_p(A) &\le \eta_1(A_1)\eta_2(A_2) \\ &\le \prod_{X \subset S_1} \theta_X(A_1)^{\nu_{1,X}} \prod_{Y \subset S_2} \theta_Y(A_2)^{\nu_{2,Y}} \\ &= \prod_{X \subset S_1} \theta_X(A)^{\nu_{1,X}} \prod_{Y \subset S_2} \theta_Y(A)^{\nu_{2,Y}}, \end{aligned}$$

where η_1 and η_2 are as before. This latter quantity is at most $\prod_{Z \subset S} \theta_Z(A)^{\nu(\{Z\})}$. This is because when $\nu(\{Z\}) \ne 0$ we have either $Z \subset S_1$ and $\nu(\{Z\}) \le \nu_1(\{Z\}) = \nu_{1,Z}$ or else $Z \subset S_2$ and $\nu(\{Z\}) \le \nu_2(\{Z\}) = \nu_{2,Z}$. $\square$

Theorem 9.2 allows one to control measures ν that are built in a certain way out of two simpler pieces. It must be pointed out that one can simply construct measures ν that are in no natural way "made from simpler pieces". For example consider a group $\mathcal{G}$ of permutations of S and the image ν of its Haar measure under the action of $\mathcal{G}$ on a given subset of S. But how do we analyze this situation, and in which direction should one look??

10. BLOCKS

Suppose we want to disprove Conjecture 9.1. Then we have to construct both the αp-spread probability measure ν and the up-class A that witness the failure of (9.2).

As we have shown in Theorem 9.2 it is not possible to construct ν by simply combining the basic examples we know. In this section we prove a "dual result": the construction of the class A itself must be fairly complicated.

A natural approach to construct complicated up-classes is by recursively combining simpler classes. Probably the simplest such method is to consider two up-classes "on disjoint blocks" and then to take either their intersection or their reunion. In particular, our previous example $A = \{X;\ \ \forall i \le m,\ \ X \cap S_i \ne \varnothing\}$ where $(S_i)_{i \le m}$ are disjoint subsets of S is of this type.

The main result of this section is that if we start with the classes $A = \{S\}$ and iterate these operations as many times as we wish the resulting class satisfies (a lot more than) Conjecture 9.1.

If we want to control a class of sets with stability properties as above, it soon turns out that it does not suffice to control these sets through their measure μ_p, but that one must also control them for *all* product measures. A more subtle fact (that seems to be deeply related with the "multiplicative form" of Conjecture 9.1) is that it is very fruitful to perform an appropriate change of scale. Given $\boldsymbol{t} = (t_i)_{i \in S}$ we denote by $\zeta_{\boldsymbol{t}}$ the product measure on $\{0,1\}^S$ such that on the i-th factor the weight of 1 is $1 - e^{-t_i}$ (and the weight of zero is e^{-t_i}). For a subset I of S we define $t_I = \prod_{i \in I} t_i$. We say that a function $\mathcal{P}(\boldsymbol{t})$ is a *polynomial* if

$$\mathcal{P}(\boldsymbol{t}) = \sum_{I \subset S} \beta_I t_I$$

where $\beta_I \geq 0$, $\beta_\varnothing = 0$. In particular a polynomial is a non-decreasing function of each of its variables.

Definition 10.1. *Given $\gamma > 0$, we say that a up-class $A \subset \{0,1\}^S$ belongs to $\mathcal{C}_\gamma$ if the following occurs. Given any $\boldsymbol{q} \in (\mathbb{R}^+)^S$, there exists a polynomial $\mathcal{P}$ (depending on $\boldsymbol{q}$) with the following properties:*

$$\forall \boldsymbol{t}, \qquad -\log(1 - \zeta_{\boldsymbol{t}}(A)) \leq \mathcal{P}(\boldsymbol{t}) \tag{10.1}$$
$$\mathcal{P}(\gamma \boldsymbol{q}) \leq -\gamma \log(1 - \zeta_{\boldsymbol{q}}(A)). \tag{10.2}$$

The notation keeps the set S implicit. Since we have assumed that a polynomial does not have a constant term, we have $\mathcal{P}(\gamma \boldsymbol{t}) \leq \gamma \mathcal{P}(\boldsymbol{t})$ for $\gamma \leq 1$; this makes it obvious that the collection $\mathcal{C}_\gamma$ increases as γ decreases.

Here is a first example.

Proposition 10.2. *The set $A = \{X \subset S \; ; \quad X \neq \varnothing\}$ belongs to class $\mathcal{C}_1$.*

Proof. We have $\zeta_{\boldsymbol{t}}(A) = 1 - \prod_{i \in S} e^{-t_i}$, so

$$-\log(1 - \zeta_{\boldsymbol{t}}(A)) = \sum_{i \in S} t_i$$

and we see that $\mathcal{P}(\boldsymbol{t}) = \sum_{i \in S} t_i$ works independently of $\boldsymbol{q}$. □

We will prove that if γ is small enough, the collection $\mathcal{C}_\gamma$ has some nice stability properties, and we will prove that a class in $\mathcal{C}_\gamma$ satisfy a lot more than Conjecture 9.1. We hope that there exists $\gamma > 0$ such that any up-class A belongs to $\mathcal{C}_\gamma$, but since this might *really* be asking for too much, we will state this as a problem rather than a conjecture.

Problem 10.3. *Is it true that there is a universal constant γ such that each up-class belongs to $\mathcal{C}_\gamma$?*

A positive solution would have sweeping consequences, since in particular it would prove Conjecture 8.5, as we show now.

Theorem 10.4. *If an up-class A belongs to $\mathcal{C}_\gamma$, and if*

$$\alpha \leq 1 - (1-p)^{\gamma/\delta}, \tag{10.3}$$

then for each δ-spread measure ν on $\{0,1\}^S$ we have

$$-\int \log(1 - \theta_{X,\alpha}(A)) d\nu(X) \leq -\gamma \log(1 - \mu_p(A)) \tag{10.4}$$

and

$$\int \theta_{X,\alpha}(A) d\nu(X) \leq \mu_p(A). \tag{10.5}$$

If (10.5) holds for each up-class A then μ_p dominates $W_\alpha(\nu)$, and this shows that a positive solution of Problem 10.3 implies Conjecture 8.5.

Proof. Consider q with $p = 1 - e^{-q}$, and $\boldsymbol{q} = (q_i)_{i \in S}$ with $q_i = q$ for each $i \in S$. By hypothesis we can find a polynomial $\mathcal{P}$ with

$$\forall \boldsymbol{t}, \quad -\log(1 - \zeta_{\boldsymbol{t}}(A)) \leq \mathcal{P}(\boldsymbol{t}) \tag{10.6}$$

$$\mathcal{P}(\gamma \boldsymbol{q}) \leq -\gamma \log(1 - \zeta_{\boldsymbol{q}}(A)) = -\gamma \log(1 - \mu_p(A)). \tag{10.7}$$

Consider the number t such that $1 - e^{-t} = \alpha$. Consider $X \subset S$, and the sequence $\boldsymbol{t} = (t_i)_{i \in S}$ given by $t_i = 0$ if $i \notin X$ and $t_i = t$ if $i \in X$. Then

$$\zeta_{\boldsymbol{t}}(A) = \theta_{X,\alpha}(A). \tag{10.8}$$

Writing $\mathcal{P}(\boldsymbol{t}) = \sum_{I \subset S} \beta_I t_I$, we see from (10.6) and (10.8) that

$$-\log(1 - \theta_{X,\alpha}(A)) \quad \leq \quad \sum_{I \subset X} \beta_I t^{\operatorname{card} I}, \tag{10.9}$$

so that for any probability measure ν we have

$$-\int \log(1 - \theta_{X,\alpha}(A)) d\nu(X) \leq \sum_I \beta_I t^{\operatorname{card} I} \nu(\{X; \quad X \supset I\})$$

and if ν is δ-spread, when $t\delta \leq \gamma q$ we get

$$-\int \log(1 - \theta_{X,\alpha}(A)) d\nu(X) \leq \sum_I \beta_I (t\delta)^{\operatorname{card} I} \leq \mathcal{P}(\gamma \boldsymbol{q}). \tag{10.10}$$

Combining with (10.6) shows that the condition $t\delta \leq \gamma q$ implies (10.4). But $t\delta \leq \gamma q$ means exactly $\alpha \leq 1 - (1-p)^{\gamma/\delta}$.

It remains only to prove (10.5). For this, using (10.6) we write that by (7.16) for any s we have

$$\zeta_{\boldsymbol{t}}(A) \leq 1 - e^{-\mathcal{P}(\boldsymbol{t})} \leq 1 - (1+s)e^{-s} + e^{-s}\mathcal{P}(\boldsymbol{t}),$$

so that using (10.8), as in (10.10) we obtain

$$\int \theta_{X,\alpha}(A) d\nu(X) \leq 1 - (1+s)e^{-s} + e^{-s}\mathcal{P}(\gamma \boldsymbol{q}).$$

We then chose $s = \mathcal{P}(\gamma \boldsymbol{q})$ and we use (10.7) to see that (since $\gamma \leq 1$)

$$1 - \exp(-\mathcal{P}(\gamma \boldsymbol{q})) \leq 1 - (1 - \mu_p(A))^\gamma \leq \mu_p(A). \qquad \square$$

Now we describe the fundamental stability property of the class $\mathcal{C}_\gamma$. Consider a partition $S = \bigcup_{i \leq n} S_i$ of S. Given up-classes $A_i \subset \{0,1\}^{S_i}$ for $i \leq n$ and an up-class $A \subset \{0,1\}^n$ we define the up-class

$$A[A_1, \ldots, A_n] = \{X \subset S; \quad \{i \leq n; \quad X \cap S_i \in A_i\} \in A\}.$$

Proposition 10.5. *If all the classes $A, A_1, \ldots, A_n$ belong to $\mathcal{C}_\gamma$ so does $A[A_1, \ldots, A_n]$.*

Proof. For $i \leq n$ and $\boldsymbol{t}_i \in (\mathbb{R}^+)^{S_i}$, let

$$F_i(\boldsymbol{t}_i) = -\log(1 - \zeta_{\boldsymbol{t}_i}(A_i)).$$

Consider $\boldsymbol{q} \in (\mathbb{R}^+)^S$, and let $\boldsymbol{q}_i$ be its projection on $(\mathbb{R}^+)^{S_i}$. Since $A_i \in \mathcal{C}_\gamma$ we can find a polynomial $\mathcal{P}_i$ with

$$\forall \boldsymbol{t}_i \in (\mathbb{R}^+)^S \;, \; F_i(\boldsymbol{t}_i) \quad \leq \quad \mathcal{P}_i(\boldsymbol{t}_i) \tag{10.11}$$
$$\mathcal{P}_i(\gamma \boldsymbol{q}_i) \quad \leq \quad \gamma F_i(\boldsymbol{q}_i). \tag{10.12}$$

For $\boldsymbol{u} \in (\mathbb{R}^+)^n$, let us consider

$$F_0(\boldsymbol{u}) = -\log(1 - \zeta_{\boldsymbol{u}}(A)).$$

Let $r_i = F_i(\boldsymbol{q}_i)$ and $\boldsymbol{r} = (r_i)_{i \le n}$. Since $A \in \mathcal{C}_\gamma$, there exists a polynomial $\mathcal{P}_0$ such that

$$\begin{aligned} \forall \boldsymbol{u} \in (\mathbb{R}^+)^n,\ F_0(\boldsymbol{u}) &\le \mathcal{P}_0(\boldsymbol{u}) && (10.13)\\ \mathcal{P}_0(\gamma \boldsymbol{r}) &\le \gamma F_0(\boldsymbol{r}). && (10.14)\end{aligned}$$

The magic formula is that for $\boldsymbol{t} \in \mathbb{R}^{+S}$,

$$F(\boldsymbol{t}) := -\log(1 - \zeta_{\boldsymbol{t}}(A[A_1, \ldots, A_n])) = F_0(\boldsymbol{u}) \quad (10.15)$$

where for $i \le n$, $u_i = F_i(\boldsymbol{t}_i)$, $\boldsymbol{u} = (u_i)_{i \le n}$ and when $\boldsymbol{t}_i$ is the projection of $\boldsymbol{t}$ on $(\mathbb{R}^+)^{S_i}$. This is a consequence of the fact that the image measure of $\zeta_{\boldsymbol{t}}$ under the map

$$X \mapsto (\mathbf{1}_{A_1}(X \cap S_1), \ldots, \mathbf{1}_{A_n}(X \cap S_n))$$

is the measure $\zeta_{\boldsymbol{u}}$, so that $\zeta_{\boldsymbol{u}}(A) = \zeta_{\boldsymbol{t}}(A[A_1, \ldots, A_n])$.

Let us define

$$\mathcal{P}(\boldsymbol{t}) = \mathcal{P}_0(\boldsymbol{v})$$

where $\boldsymbol{v} = (v_i)$, $v_i = \mathcal{P}_i(\boldsymbol{t}_i)$. This is a polynomial. Combining (10.15) with (10.11) and (10.13) we get

$$\begin{aligned} \forall \boldsymbol{t}\ ,\ F(\boldsymbol{t}) &= F_0(F_1(\boldsymbol{t}_1), \ldots, F_n(\boldsymbol{t}_n))\\ &\le \mathcal{P}_0(\mathcal{P}_1(\boldsymbol{t}_1), \ldots, \mathcal{P}_n(\boldsymbol{t}_n)) = \mathcal{P}(\boldsymbol{t}).\end{aligned}$$

Combining (10.12), (10.14) and (10.15) we obtain

$$\begin{aligned} \mathcal{P}(\gamma \boldsymbol{q}) &= \mathcal{P}_0(\mathcal{P}_1(\gamma \boldsymbol{q}_1), \ldots, \mathcal{P}_n(\gamma \boldsymbol{q}_n))\\ &\le \mathcal{P}_0(\gamma F_1(\boldsymbol{q}_1), \ldots, \gamma F_n(\boldsymbol{q}_n))\\ &= \mathcal{P}_0(\gamma r_1, \ldots, \gamma r_n)\\ &\le \gamma F_0(r_1, \ldots, r_n)\\ &= \gamma F_0(F_1(\boldsymbol{q}_1), \ldots, F_n(\boldsymbol{q}_n)) = \gamma F(\boldsymbol{q}). \quad \square\end{aligned}$$

The true difficulty is to prove that any class at all other than the classes of Proposition 10.2 belongs to $\mathcal{C}_\gamma$.

THEOREM 10.6. *There exists $\gamma > 0$ such that for each S the class $A = \{S\}$ belongs to the class $\mathcal{C}_\gamma$.*

Combining Propositions 10.2 et 10.5 with Theorem 10.6 we see that starting with the classes of Theorem 10.6, and iterating the operations "union on disjoint blocks" and "intersection on disjoint blocks" we can create only classes in $\mathcal{C}_\gamma$.

We define the function

$$F(u, v) = -\log(e^{-u} + e^{-v} - e^{-u-v})$$

for $u, v \ge 0$.

To prove Theorem 10.6 it suffices using Proposition 10.5 to consider the case where $\text{card}S = 2$, in which case Theorem 10.6 is equivalent to the following.

THEOREM 10.7. *There exists a number $\gamma > 0$ with the following property. Given $u_0, v_0 \ge 0$, we can find $a, b, c \ge 0$ such that*

$$\begin{aligned} \forall u, v\ ,\ F(u, v) &\le au + bv + cuv && (10.16)\\ au_0 + bv_0 + \gamma c u_0 v_0 &\le F(u_0, v_0). && (10.17)\end{aligned}$$

It would be nice to know what is the best (= largest) value of γ for which Theorem 10.6 holds, and in particular whether it holds for $\gamma = 1$. Our present argument does not allow this.

In order to prove Theorem 10.6 we have to gain some understanding of which triples a, b, c satisfy (10.16). Fixing v, we observe that

$$\frac{\partial}{\partial u} F(u, v) = \frac{e^{-u}(1 - e^{-v})}{e^{-u} + e^{-v} - e^{-u-v}} \le 1 - e^{-v}.$$

Thus, since $F(0, v) = 0$, if $1 - e^{-v} \le a + cv$, then (10.16) holds for each u, and we need only be concerned with the case $a + cv \le 1 - e^{-v}$. Assuming that this occurs for certain values of v, these form an interval $[v_1, v_2]$. Consider $v_1 \le v \le v_2$. Setting $w = a + cv$, in order for (10.16) to hold we need that

$$\max_u (F(u, v) - uw) \le bv. \quad (10.18)$$

The maximum in the right-hand side is obtained for $\partial F / \partial u(u, v) = w$, i.e.

$$e^{-u} = \frac{we^{-v}}{(1 - w)(1 - e^{-v})}, \quad (10.19)$$

and after a few lines of algebra this maximum is found to be

$$\begin{aligned} \max_u (F(u, v) - uw) = f(v) &:= (1 - w)\log(1 - w)\\ &+ v(1 - w) + w\log w - w\log(1 - e^{-v}). \end{aligned} \quad (10.20)$$

This function is defined on the interval $[v_1, v_2]$. If $v = v_j$ for $j = 1$ or $j = 2$ one has $w = 1 - e^{-v_j}$ so one sees from (10.20) that $f(v_1) = f(v_2) = 0$. Also, from (10.20) we get (recalling that $w = a + cv$)

$$\begin{aligned} f'(v) &= 1 - \frac{w}{1 - e^{-v}}\\ &+ c(\log w - \log(1 - w) - v - \log(1 - e^{-v}))\\ &= 1 - \frac{w}{1 - e^{-v}} + c\left(\log \frac{we^{-v}}{(1 - w)(1 - e^{-v})}\right).\end{aligned}$$

The last term is ≤ 0 by (10.19), so that

$$f'(v) \le 1 - w = 1 - (a + cv). \quad (10.21)$$

LEMMA 10.8. *If*

$$(1 - a - b)^2 \le c(1 - a)v_1 \quad (10.22)$$

then (10.16) *holds.*

Proof. We have to prove (10.18) i.e. that $f(v) \le bv$ for $v_1 \le v \le v_2$. We have $f(v_1) = 0$, and (10.21) yields $f'(v) \le b$ if $1 - (a + cv) \le b$, i.e. $v \ge (1 - b - a)/c$. Thus it suffices to show that $f(v) \le bv$ for $v \le (1 - b - a)/c$. But since $f(v_1) = 0$ and since $f'(v) \le 1 - a$ by (10.21) we have

$$f(v) \le (1 - a)(v - v_1).$$

Now

$$(1 - a)(v - v_1) \le bv$$

since this inequality is equivalent to $(1 - a - b)v \le (1 - a)v_1$ and using (10.22) and since $v \le (1 - b - a)/c$. $\square$

Proof of Theorem 8.7. Without loss of generality we assume that $v_0 \ge u_0$, and we set $t = v_0 - u_0$. Since log(1 +

$x) \le x$ we have

$$\begin{aligned} F(u_0, v_0) &= -\log(e^{-u_0} + e^{-u_0-t} - e^{-2u_0-t}) \\ &= u_0 - \log(1 + e^{-t}(1 - e^{-u_0})) \\ &\ge u_0 - e^{-t}(1 - e^{-u_0}). \end{aligned}$$

Instead of (10.17) we will achieve the better inequality

$$au_0 + bv_0 + \gamma c u_0 v_0 \le u_0 - e^{-t}(1 - e^{-u_0}). \tag{10.23}$$

The proof totally lacks of glory. We produce the values of a, b, c by pure fiat depending on the values of u_0 and t, and we check in each case that (10.22) and (10.23) hold.

Case 1: $t \ge u_0$. If $b = 0$, $c = e^{-s}$, $a = 1 - (1+s)e^{-s}$ for a certain number s, then (10.16) is automatic by (7.16), and

$$au_0 + bv_0 + \gamma c u_0 v_0 = (1 - (1+s)e^{-s})u_0 + \gamma e^{-s} u_0 v_0$$

is minimum for $s = \gamma v_0$, and this minimum has value $u_0(1 - e^{-\gamma v_0})$. And if $\gamma \le 1/2$ we have

$$u_0(1 - e^{-\gamma v_0}) \le u_0 - e^{-t}(1 - e^{-u_0})$$

because $1 - e^{-u_0} \le u_0$ and $t = v_0 - u_0 \ge v_0/2 \ge \gamma v_0$ since $2u_0 \le u_0 + t = v_0$. Thus (10.23) holds in this case.

Case 2: $t \le u_0 \le 4$. In that case, $u_0 \le 4$ and $v_0 \le 2u_0$. For a certain number L we have $e^{-u_0} \ge 1 - u_0 + u_0^2/L$ and

$$u_0 - e^{-t}(1 - e^{-u_0}) \ge u_0 - e^{-t}(u_0 - \frac{u_0^2}{L}) \ge \frac{e^{-4}u_0^2}{L} \ge \frac{e^{-4}u_0 v_0}{2L}.$$

Thus (assuming $\gamma \le e^{-4}/2L$) we simply take $a = b = 0$, $c = 1$ to obtain (10.23) and we note that $1 - e^{-v} \le a + cv = v$, so that (10.16) holds automatically.

Case 3: $t \le u_0$, $u_0 \ge 4$. We chose

$$c = \frac{1}{2\gamma}\frac{e^{-t}}{u_0 v_0}\ ;\ b = \frac{e^{-t}}{4\max(1,t)}\ ;\ a + b = 1 - \frac{2e^{-t}}{u_0}.$$

We then have

$$au_0 + bv_0 = (a+b)u_0 + tb \le u_0 - 2e^{-t} + \frac{e^{-t}}{4},$$

and consequently

$$au_0 + bv_0 + \gamma c u_0 v_0 \le u_0 - e^{-t},$$

so it is obvious that (10.23) holds. To prove (10.16), we check that if γ is small enough then (10.22) holds, and since $1 - a \ge b$ it suffices to show that $(1 - a - b)^2 \le bcv_1$, i.e.

$$\frac{4e^{-2t}}{u_0^2} \le \frac{e^{-2t}}{8\gamma u_0 v_0 \max(1,t)} v_1. \tag{10.24}$$

Since v_1 satisfies $a + cv_1 = 1 - e^{-v_1}$, we have

$$e^{-v_1} \le 1 - a = b + \frac{2e^{-t}}{u_0} \le \frac{3}{4}e^{-t}$$

since $u_0 \ge 4$, and therefore $v_1 \ge \max(1,t)/L_1$. Since $v_0 = u_0 + t \le 2u_0$, this implies (10.24) when $\gamma = 1/(64L_1)$. □

Problem 10.9. *Does there exist $\gamma > 0$ such that all the classes*

$$A = \{X \subset S;\ \ \mathrm{card}X \ge k\}$$

belong to $\mathcal{C}_\gamma$?

11. MISCELLANEOUS PROOFS

To lighten notations, we write μ rather that μ_p.

Theorem 11.1. *There exists a universal constant L_0 with the following property. Consider any set S, any numbers $t_i \ge 0$, $i \in S$, and any $0 < p < 1/2$. Then for $u \ge L_0$ we can find a family $\mathcal{I}$ of subsets of S with the following properties:*

$$\sum_{i\in X} t_i \ge L_0 up \sum_{i\in S} t_i \Rightarrow \exists I \in \mathcal{I},\ I \subset X \tag{11.1}$$

$$\sum_{I\in\mathcal{I}} p^{\mathrm{card} I} \le 4\mu_p\left(\left\{X;\ \sum_{i\in X} t_i \ge pu \sum_{i\in S} t_i\right\}\right). \tag{11.2}$$

In particular, taking $u = \max(8, L_0)$ this proves Conjecture 5.7 in the case where T is reduced to a single point, but in fact it does a little bit more, and yields the following (very weak) support of Conjecture 5.7.

Proposition 11.2. *Conjecture 5.7 holds when T consists of elements with disjoint support.*

Proof. Let us enumerate the elements of T as $\boldsymbol{t}^1, \ldots, \boldsymbol{t}^k$, let $U = \mathsf{E}_p \sup_{\boldsymbol{t}\in T} \sum_{i\in X} t_i$ and let

$$A_k = \left\{X;\ \sum_{i\in X} t_i^k \ge L_0 U\right\}.$$

Thus

$$\mu\left(\bigcup_{k\le k_0} A_k\right) \le \mu\left(\sup_{\boldsymbol{t}\in T}\sum_{i\in X} t_i \ge L_0 U\right) \le \frac{1}{L_0}.$$

Since the classes A_k are independent for μ_p (because the sequences $\boldsymbol{t}^k$ have disjoint supports) we have

$$\mu\left(\bigcup_{k\le k_0} A_k\right) = 1 - \prod_{k\le k_0}(1 - \mu(A_k))$$

so that

$$\prod_{k\le k_0}(1 - \mu(A_k)) \ge 1 - \frac{1}{L_0}.$$

Assuming without loss of generality that L_0 is large enough, we get

$$\sum_{k\le k_0} -\log(1 - \mu(A_k)) \le -\log\left(1 - \frac{1}{L_0}\right) \le \frac{1}{8}$$

and since $x \le -\log(1-x)$ we have

$$\sum_{k\le k_0} \mu(A_k) \le \frac{1}{8}.$$

Now, for each k, we have $U \ge U_k := p\sum_{i\in S} t_i^k$. We use Theorem 11.1 with $u = L_0 U/U_k \ge L_0$ to see that if

$$B_k = \left\{X;\ \sum_{i\in X} t_i^k \ge L_0^2 U\right\},$$

then we can find a family $\mathcal{I}_k$ of subsets of S with $B_k \subset \bigcup_{I\in\mathcal{I}_k} H_I$ and $\sum_{I\in\mathcal{I}_k} p^{\mathrm{card} I} \le 4\mu(A_k)$. The family $\mathcal{I} = \bigcup_k \mathcal{I}_k$ satisfies

$$\left\{X;\ \sup_{\boldsymbol{t}\in T}\sum_{i\in X} t_i \ge L_0^2 U\right\} = \bigcup_k B_k \subset \bigcup_{I\in\mathcal{I}} H_I$$

and $\sum_{I\in\mathcal{I}} p^{\mathrm{card} I} \le 1/2$. $\square$

Proof of Theorem 11.1. We set

$$U = p\sum_{i\in S} t_i \tag{11.3}$$

and given $u \ge 2$ we consider

$$B = \left\{X;\ \sum_{i\in X} t_i \ge uU\right\} \tag{11.4}$$

so that

$$\mu(B) \le \frac{1}{u}. \tag{11.5}$$

We consider a number $w \ge u$ to be specified later and

$$A = \left\{X;\ \sum_{i\in X} t_i \ge 8wU\right\}. \tag{11.6}$$

Consider

$$S_0 = \{i \in S;\ t_i \ge uU\},$$

and the family $\mathcal{I}_0$ consisting of the sets $I \subset S_0$ with $\mathrm{card} I = 1$. Thus

$$X \cap S_0 \ne \varnothing \Rightarrow \exists I \in \mathcal{I}_0,\ I \subset X \tag{11.7}$$

$$\sum_{I\in\mathcal{I}_0} p^{\mathrm{card} I} = p\,\mathrm{card}\mathcal{I}_0 = p\,\mathrm{card} S_0. \tag{11.8}$$

We have

$$X \cap S_0 \ne \varnothing \Rightarrow \left\{\sum_{i\in X} t_i \ge uU\right\} \Rightarrow X \in B$$

so, if $M_0 = \mathrm{card} S_0$,

$$1 - (1-p)^{M_0} \le \mu(B) \le \frac{1}{u}$$

and

$$e^{-pM_0} \ge (1-p)^{M_0} \ge 1 - \mu(B),$$

so that, since $u \ge 2$ and therefore $\mu(B) \le 1/2$,

$$pM_0 \le -\log(1-\mu(B)) \le 2\mu(B)$$

and (11.8) implies

$$\sum_{I\in\mathcal{I}_0} p^{\mathrm{card} I} \le 2\mu(B). \tag{11.9}$$

Consider

$$A' = \left\{X;\ X \cap S_0 = \varnothing,\ \sum_{i\in X} t_i \ge 8wU\right\}.$$

We have controlled $A \setminus A'$ through (11.7) and (11.9) and we turn to the control of A'.

For $k \ge 0$, we set

$$J_k = \{i \in S;\ 4^{-k-1}uU \le t_i < 4^{-k}uU\} \tag{11.10}$$

and $M_k = \mathrm{card} J_k$. Given $X \in A'$ we set $n_k = \mathrm{card}(J_k \cap X)$. Let

$$W_1 = \{k \ge 0;\ n_k \le wpM_k\}.$$

Thus

$$\begin{aligned}\sum_{k\in W_1} uU4^{-k} n_k &\le wp\sum_k uU4^{-k}M_k \\ &\le 4wp\sum_k \sum_{i\in J_k} t_i \le 4wU. \end{aligned}\tag{11.11}$$

Let

$$W_2 = \left\{k \ge 0;\ n_k \le 2^k\frac{w}{u}\right\}$$

so that

$$\sum_{k\in W_2} uU4^{-k}n_k \le 2wU. \tag{11.12}$$

Let $W = \{k \ge 0;\ k \notin W_1 \cap W_2\}$. Since $X \in A'$ we have

$$\begin{aligned}8wU &\le \sum_{i\notin S_0, i\in X} t_i \le \sum_{k\ge 0} 4^{-k}uU\mathrm{card}(X\cap J_k) \\ &\le \sum_{k\ge 0} uU4^{-k}n_k\end{aligned}$$

and we deduce from (11.11) and (11.9) that

$$\sum_{k\in W} uU4^{-k}n_k \ge 2wU$$

i.e.

$$\sum_{k\in W} 4^{-k}n_k \ge 2\frac{w}{u}. \tag{11.13}$$

For a set W of integers and $\boldsymbol{n} = (n_k)_{k\in W}$ let us define

$$A_{W,\boldsymbol{n}} = \{X \subset S;\ \forall k \in W,\ \mathrm{card}(X\cap J_k) = n_k\}.$$

We have shown that

$$A' \subset \bigcup_{W,\boldsymbol{n}} A_{W,\boldsymbol{n}},$$

where the union is over all choices of W and of $\boldsymbol{n}$ such that (11.13) holds together with

$$n_k > 2^k\frac{w}{u}\quad;\quad n_k > wpM_k \tag{11.14}$$

To control the classes $A_{W,\boldsymbol{n}}$, we will now compare them with the classes

$$B_{W,\boldsymbol{n}} = \left\{X \subset S;\ \forall k \in W,\ \mathrm{card}(X\cap J_k) \ge \frac{2u}{w}n_k\right\}.$$

First we note that by (11.13) for X in B we have

$$\sum_{k\in W} 4^{-k}\mathrm{card}(X\cap J_k) \ge 4$$

so that by (11.10)

$$\sum_{i\in X} t_i \ge uU$$

and thus

$$B_{W,\boldsymbol{n}} \subset B. \tag{11.15}$$

We have

$$\mu(B_{W,\boldsymbol{n}}) = \prod_{k\in W} T(k) \tag{11.16}$$

where

$$T(k) = \mu(\{X \subset S;\ \mathrm{card}(X\cap J_k) \ge m_k\})$$

and where m_k is the smallest integer $\geq 2un_k/w$. Thus

$$T(k) \geq p^{m_k}(1-p)^{M_k-m_k}\binom{M_k}{m_k}.$$

We first show that in this bound the factor $(1-p)^{M_k-m_k}$ is not dangerous. Since $u \geq 2$, by (11.14) we have $m_k \geq 2un_k/w \geq 4pM_k$ and thus

$$(1-p)^{M_k} \geq \left(\frac{1}{L}\right)^{m_k}.$$

Here, as well as in the rest of the paper, L denotes a number, that need not be the same at each occurrence, while $L_0, L_1, \cdots$ denote specific constants. Thus, from (5.6) we get

$$T(k) \geq \left(\frac{pM_k}{Lm_k}\right)^{m_k}$$

and from (11.15) and (11.16) that

$$\prod_{k\in W}\left(\frac{pM_k}{L_1 m_k}\right)^{m_k} \leq \mu(B). \tag{11.17}$$

Consider now the class $\mathcal{I}_{W,\boldsymbol{n}}$ of subsets I of S that satisfy $I \subset \bigcup_{k\in W} J_k$ and $\operatorname{card}(I\cap J_k) = n_k$ for $k \in W$. Then, obviously,

$$\forall X \in A_{W,\boldsymbol{n}},\ \exists I \in \mathcal{I}_{W,\boldsymbol{n}},\ I \subset X. \tag{11.18}$$

On the other hand

$$\begin{aligned}\sum_{I\in\mathcal{I}_{W,\boldsymbol{n}}} p^{\operatorname{card} I} &= \prod_{k\in W} p^{n_k}\binom{M_k}{n_k}\\ &\leq \prod_{k\in W}\left(\frac{L_2 pM_k}{n_k}\right)^{n_k} \qquad (11.19)\end{aligned}$$

using (5.6) again. The idea is now to use (11.17) to control the sum of the quantities (11.19) over all choices of W and $\boldsymbol{n}$ when $w = 8L_1L_2u$ (it is here that we fix w). Since m_k is the smallest integer $\geq 2un_k/w > 2^{k+1} \geq 2$, we have $m_k \leq 4un_k/w$, so $n_k \geq wm_k/4u \geq 2L_1L_2um_k$. Therefore

$$\frac{L_2pM_k}{n_k} \leq \frac{pM_k}{2L_1m_k},$$

and

$$\left(\frac{L_2pM_k}{n_k}\right)^{n_k} \leq 2^{-n_k}\left(\frac{pM_k}{L_1m_k}\right)^{n_k}.$$

Using the second part of (11.14) we have $pM_k \leq n_k/w \leq m_k/2u \leq L_1m_k$ and since $n_k \geq 2L_1L_2m_k \geq m_k$, we have

$$\left(\frac{L_2pM_k}{n_k}\right)^{n_k} \leq 2^{-n_k}\left(\frac{pM_k}{L_1m_k}\right)^{n_k} \leq 2^{-n_k}\left(\frac{pM_k}{L_1m_k}\right)^{m_k}.$$

Thus combining (11.17) and (11.19) we get

$$\sum_{I\in\mathcal{I}_{W,\boldsymbol{n}}} p^{\operatorname{card} I} \leq \left(\frac{1}{2}\right)^{\sum_{k\in W} n_k}\mu(B)$$

and it suffices to show that if we sum over all choices of W and n_k the coefficients $(1/2)^{\sum_{k\in W} n_k}$ the result is ≤ 2. This sum is at most

$$\prod_{k\geq 0}\Bigl(1 + \sum_{n_k > 2^k w/u} 2^{-n_k}\Bigr).$$

Now, whenever $w/u \geq 4$, we have

$$\sum_{n_k>2^k w/u} 2^{-n_k} \leq 2^{-2^{k+2}}$$

and $\prod_{k\geq 0}(1 + 2^{-2^{k+2}}) \leq 2$. □

Our next result also provides (much deeper) support for Conjecture 5.7.

THEOREM 11.3. *There exists a number q with the following property. Consider a set S that is a disjoint union $S = \bigcup_{1\leq k\leq k_0} S_k$, and $A \subset \{0,1\}^S$ such that*

$$\forall J \in A,\ \operatorname{card}(J\cap S_k) = 2^k. \tag{11.20}$$

Assume that there exists a p-spread probability measure ν with $\nu(A) = 1$. Then given any class $B \subset \{0,1\}^S$ with $\mu(B) \geq 1 - 1/q$, we can find $J^1, \dots, J^q$ in B and $J \in A$ such that

$$\forall k \leq k_0,\ \operatorname{card}((J \setminus (J^1\cup\cdots\cup J^q))\cap S_k) \leq 2^{k-1}. \tag{11.21}$$

The following consequence should be compared to Conjecture 6.8

COROLLARY 11.4. *There exists a number L with the following property. Consider any set S that is a disjoint union $S = \bigcup_{1\leq k\leq k_0} S_k$, and consider a class T of sequences $\boldsymbol{t} = (t_i)_{i\in S}$ with $t_i \geq 0$. Assume that for each $\boldsymbol{t} \in T$ and each k, we have $\operatorname{card}\{i \in S_k;\ t_i \neq 0\} = 2^k$, and that for $i, j \in S_k$, if $t_i \neq 0$, $t_j \neq 0$ then $t_i = t_j$. (That is, each sequence $(t_i)_{i\in S_k}$ takes only two possible values, one of which is 0). Assume there is a probability measure ν on T such that*

$$\forall I \subset S,\ \nu(\{\boldsymbol{t};\ i\in I \Rightarrow t_i > 0\}) \leq p^{\operatorname{card} I}. \tag{11.22}$$

Then

$$\mathsf{E}_p \sup_{\boldsymbol{t}\in T}\sum_{i\in X} t_i \geq \frac{1}{L}\inf_{\boldsymbol{t}\in T}\sum_{i\in S} t_i. \tag{11.23}$$

Proof. Given $\boldsymbol{t} \in T$ let

$$J_{\boldsymbol{t}} = \{i \in S; t_i \neq 0\}.$$

Let $A \subset \{0,1\}^S$ be the class of the sets $J_{\boldsymbol{t}}$ as $\boldsymbol{t}$ varies in T and let ν' be the image of ν under the map $\boldsymbol{t} \mapsto J_{\boldsymbol{t}}$. By construction $\nu'(A) = 1$ and by (11.22) ν' is p-spread.

Let us now set

$$U = \mathsf{E}_p \sup_{\boldsymbol{t}\in T}\sum_{i\in X} t_i$$

and consider the class

$$B = \Bigl\{X\ ;\ \sup_{\boldsymbol{t}\in T}\sum_{i\in X} t_i \leq qU\Bigr\},$$

so that by Markov inequality we have $\mu(B) \geq 1 - 1/q$. We then apply Theorem 11.3 to find a set $J = J_{\boldsymbol{t}} \in A$ and sets $J^1, \dots, J^q \in B$ such that if $J' = J^1\cup\cdots\cup J^q$ we have

$$\forall k \leq k_0\ ,\ \operatorname{card}(J_{\boldsymbol{t}} \setminus J') \leq 2^{k-1}.$$

Since in each set S_k the sequence $\boldsymbol{t}$ takes only two values, say 0 and a_k, we have

$$\sum_{i\in S_k\cap(J_{\boldsymbol{t}}\cap J')} t_i = a_k\operatorname{card}(J_{\boldsymbol{t}}\cap J') \geq a_k 2^{k-1} = \frac{1}{2}\sum_{i\in S_k} t_i.$$

By summation over k we see that

$$\sum_{i\in J'} t_i \geq \frac{1}{2}\sum_i t_i.$$

Consequently since $J' = J^1 \cup \ldots \cup J^q$, then for some $\ell \leq q$ we have

$$\sum_{i\in J^\ell} t_i \geq \frac{1}{q}\sum_{i\in J'} t_i \geq \frac{1}{2q}\inf_{t\in T}\sum_{i\in S} t_i$$

and since $J^\ell \in B$ the definition of B shows that $\sum_{i\in J^\ell} t_i \leq qU$, so that

$$U = \mathsf{E}_p \sup_{\boldsymbol{t}\in T}\sum_{i\in X} t_i \geq \frac{1}{2q^2}\inf_{t\in T}\sum_{i\in S} t_i,$$

which completes the proof. □

The proof of Theorem 11.3 elaborates on one idea of [9] . It combines isoperimetry with the second moment method. One version of the second moment method is the observation that for a r.v. $Z \geq 0$ we have

$$\mathsf{P}(Z > 0) \geq \frac{(\mathsf{E}Z)^2}{\mathsf{E}(Z^2)}. \tag{11.24}$$

This will used as follows. Suppose that to each $J \in A$ we associate a set $G_J \subset \{0,1\}^S$ and for $X \subset S$ let

$$Z(X) = \nu(\{J;\ X \in G_J\}).$$

Then the following holds true.

$$\{Z > 0\} \subset \bigcup_{J\in A} G_J \tag{11.25}$$

$$\begin{aligned}\mathsf{E}_p Z &= \int \nu(\{J;\ X\in G_J\})d\mu(X)\\ &= \int \mu(G_J)d\nu(J)\end{aligned}$$

$$\begin{aligned}\mathsf{E}_p Z^2 &= \int \nu(\{J;\ X\in G_J\})^2 d\mu(X)\\ &= \int \nu^{\otimes 2}(\{(J,J');\ X\in G_J\cap G_{J'}\})d\mu(X)\\ &= \int \mu(G_J\cap G_{J'})d\nu(J)d\nu(J').\end{aligned}$$

Thus, combining (11.24) and (11.25)

$$\mu\left(\bigcup_{J\in A} G_J\right) \geq \frac{\left(\int \mu(G_J)d\nu(J)\right)^2}{\int \mu(G_J\cap G_{J'})d\nu(J)d\nu(J')}. \tag{11.26}$$

To make this relation useful we need lower bounds on the denominator. Let us assume that for some subset S' of S we have

$$\forall J\in A, \qquad \mathrm{card} J\cap S' = m \tag{11.27}$$

$$\forall J\in A \qquad G_J \subset H_{J\cap S'}. \tag{11.28}$$

Then $\mathrm{card}(J\cap S')\cup(J'\cap S')) = 2m - \mathrm{card}(J\cap J'\cap S)$ so that

$$\begin{aligned}\mu(G_J\cap G_{J'}) &\leq \mu(H_{J\cap S'}\cap H_{J'\cap S'})\\ &= p^{2m-\mathrm{card}(J\cap J'\cap S)}.\end{aligned}$$

Thinking of J as fixed, we have

$$\begin{aligned}\mu(G_J\cap G_{J'}) > p^{2m-k+1} &\Rightarrow \mathrm{card}(J\cap J'\cap S)\geq k\\ &\Rightarrow \exists I\subset J,\ \mathrm{card} I = k,\ J'\supset I\end{aligned}$$

and thus

$$\begin{aligned}\nu(\{J';\ \mu(G_J\cap G_{J'})\geq p^{2m-k+1}\}) &\leq \sum_{I\subset J, \mathrm{card} I = k}\nu(H_I)\\ &\leq \binom{m}{k}p^k\end{aligned}$$

because $\nu(H_I)\leq p^{\mathrm{card} I}$ since ν is p-spread. It follows that

$$\int \mu(G_J\cap G_{J'})d\nu(J') \leq p^{2m}\sum_k \binom{m}{k} = 2^m p^{2m}.$$

So we have proved the following.

Lemma 11.5. *Under* (11.27) *and* (11.28) *and if ν is p-spread we have*

$$\mu\left(\bigcup_{J\in A} G_J\right) \geq 2^{-m}p^{-2m}\left(\int \mu(G_J)d\nu(J)\right)^2. \tag{11.29}$$

We are now going to formulate the main step of the proof of Theorem 11.3. For this it helps to think of $\{0,1\}^S$ as the product of the spaces $\{0,1\}^{S_\ell}$, $\ell = 1,\ldots,k_0$. We denote by μ_ℓ the measure μ_{p,S_ℓ}.

Proposition 11.6. *For $\ell = 1,\ldots,k_0$ we can find classes $C_\ell \subset \{0,1\}^{S_1}\times\cdots\times\{0,1\}^{S_\ell}$ with the following properties*

$$\mu_1(C_1) \geq \frac{1}{16} = 2^{-4} \tag{11.30}$$

$$\begin{aligned}&\forall \ell < k_0,\ \forall J\in C_\ell,\\ &\mu_{\ell+1}(\{Y\in\{0,1\}^{S_{\ell+1}};\ J\cup Y\in C_{\ell+1}\}) \geq 2^{-2^{\ell+3}}\end{aligned} \tag{11.31}$$

$$C_{k_0} = \{Y\subset S;\ \exists J\in A,\ J\subset Y\}. \tag{11.32}$$

Proof. We define C_{k_0} by (11.32) and by decreasing induction the classes C_ℓ as the largest such that (11.31) holds. Therefore,

$$\begin{aligned}&\forall \ell < k_0,\ \forall J\notin C_\ell,\\ &\mu_{\ell+1}(\{Y\in\{0,1\}^{S_{\ell+1}};\ J\cup Y\in C_{\ell+1}\}) < 2^{-2^{\ell+3}}.\end{aligned} \tag{11.33}$$

The issue is then to prove (11.30). We proceed by contradiction, assuming that $\mu_1(C_1) < 1/16$. Let

$$C_1' = C_1\times\{0,1\}^{S_2}\times\cdots\times\{0,1\}^{S_{k_0}}$$

and for all $J\in A$ let

$$G_J^1 = C_1'\cap H_{J\cap S_1}.$$

We can use (11.29) with $m = 2$ because $\mathrm{card}(J\cap S_1) = 2$ so that

$$\begin{aligned}\frac{1}{16} > \mu_1(C_1) &= \mu(C_1') \geq \mu(\bigcup_{J\in A} G_J^1)\\ &\geq \frac{1}{4}p^{-4}\left(\int \mu(G_J^1)d\nu(J)\right)^2\end{aligned}$$

and thus

$$\int \mu(G_J^1)d\nu(J) \leq \frac{p^2}{2}.$$

Since $\mu_1(H_{J\cap S_1}) = p^2$, if we set

$$D_1 = \{0,1\}^S \setminus C'_1 = \{X \subset S;\ X \cap S_1 \notin C_1\}$$

we then have proved that

$$\int \mu(D_1 \cap H_{J\cap S_1}) d\nu(J) \ge \frac{p^2}{2}. \qquad (11.34)$$

For $J \in A$ consider now the classes

$$D_1 \cap H_{J\cap(S_1\cup S_2)}.$$

Since $\mathrm{card} J \cap S_2 = 2^2 = 4$ for $J \in A$, and since the fact that $X \in D_1$ or not is determined by the set $X \cap S_1$ we have

$$\mu(D_1 \cap H_{J\cap(S_1\cup S_2)}) = p^4 \mu(D_1 \cap H_{J\cap S_1})$$

and by (11.34) we have

$$\int \mu(D_1 \cap H_{J\cap(S_1\cup S_2)}) d\nu(J) \ge \frac{p^6}{2}. \qquad (11.35)$$

Consider the class

$$D'_1 = \{X \subset S;\ X \cap S_1 \notin C_1,\ X \cap (S_1 \cup S_2) \in C_2\}.$$

Using (11.33) for $\ell = 1$ and Fubini Theorem, we see that

$$\mu(D'_1) \le 2^{-16}.$$

Consider the classes

$$G^2_J = D'_1 \cap H_{J\cap(S_1\cup S_2)}$$

so that we can use (11.29) with $m = 6$ to get

$$2^{-16} \ge \mu(D'_1) \ge \mu\left(\bigcup G^2_J\right) \ge 2^{-6} p^{-12} \left(\int \mu(G^2_J) d\nu(J)\right)^2$$

and thus

$$\int \mu(G^2_J) d\nu(J) \le \frac{1}{2^4} p^6. \qquad (11.36)$$

Let $D_2 = D_1 \setminus D'_1$. Comparison of (11.36) and (11.35) yields

$$\int \mu(D_2 \cap H_{J\cap(S_1\cup S_2)}) d\nu(J) \ge \frac{p^6}{4},$$

while

$$D_2 = \{X \subset S;\ X \cap S_1 \notin C_1,\ X \cap (S_1 \cup S_2) \notin C_2\}.$$

Proceeding in this manner we show by induction on $\ell \le k_0$ that

$$\int \mu(D_\ell \cap H_{J\cap(S_1\cup S_2\cup\cdots\cup S_\ell)}) d\nu(J) \ge \frac{p^{2^{\ell+1}-2}}{2^\ell} \qquad (11.37)$$

where

$$D_\ell = \{X \subset S;\ X \cap S_1 \notin C_1, \ldots, X \cap (S_1 \cup \cdots \cup S_\ell) \notin C_\ell\}.$$

But for $\ell = k_0$ this is a contradiction because $D_{k_0} \cap C_{k_0} = \varnothing$ while for $J \in A$ we have $H_J \subset C_{k_0}$ by definition of C_{k_0}, so that $\mu(D_{k_0} \cap H_J) = 0$ and the integrand in (11.37) is 0. □

Lemma 11.7. *Consider numbers b_ℓ, $\ell = 1, \ldots, k_0$ with $b_\ell < 1$. Consider a class $B \subset \{0,1\}^S$ with $\mu(B) \ge 1 - \prod_{1\le\ell\le k_0}(1 - b_\ell)$. Then we can find classes $B_\ell \subset \{0,1\}^{S_1} \times \cdots \times \{0,1\}^{S_\ell}$ with $B = B_{k_0}$ and*

$$\mu(B_1) \ge b_1 \qquad (11.38)$$

$$\forall \ell < k_0,\ \forall J \in B_\ell,$$
$$\mu_{\ell+1}(\{Y \in \{0,1\}^{S_{\ell+1}};\ J \cup Y \in B_{\ell+1}\}) \ge b_{\ell+1}. \qquad (11.39)$$

Proof. Starting with $B_{k_0} = B$ we construct the classes B_ℓ by decreasing induction over ℓ, these classes being as large as possible such that (11.39) holds. Denoting by μ'_ℓ the measure $\mu_1 \otimes \cdots \otimes \mu_\ell$ on $\{0,1\}^{S_1} \times \cdots \times \{0,1\}^{S_\ell}$, we see by Fubini theorem that

$$b_{\ell+1}(1 - \mu'_\ell(B_\ell)) + \mu'_\ell(B_\ell) \ge \mu'_{\ell+1}(B_{\ell+1})$$

so that

$$(1 - b_{\ell+1})(1 - \mu'_\ell(B_\ell)) \le 1 - \mu'_{\ell+1}(B_{\ell+1})$$

and by induction

$$1 - \mu'_\ell(B_\ell) \le \frac{1}{\prod_{\ell+1\le\ell'\le k_0}(1 - b_{\ell'})}(1 - \mu(B))$$

which implies (11.38) when $\ell = 1$. □

The other key ingredient in the proof of Theorem 11.3 is the following "isoperimetric" result [7]. We mention here only the form we need.

Lemma 11.8. *Consider two integers q and n. Then for any classes $A_1 \ldots, A_q \subset \{0,1\}^S$ we have*

$$\begin{aligned}\mu(\{X \subset S;\ &\forall I^1 \in A_1, \ldots, I^q \in A_q,\\ &\mathrm{card}(X \setminus (I^1 \cup \cdots \cup I^q)) \ge n\})\\ &\le \frac{1}{\mu(A_1)\cdots\mu(A_q)} \frac{1}{q^n}.\end{aligned} \qquad (11.40)$$

Proof of Theorem 11.3. Let $q = 2^{2^6}$, and for $\ell \ge 1$ let $b_\ell = 2^{-2^\ell/q}$ so that for $\ell \ge 1$ we have

$$\left(\frac{1}{b_\ell}\right)^q \frac{1}{q^{2^{\ell-1}}} < 2^{-2^{\ell+4}}. \qquad (11.41)$$

Let $b = 1 - \prod_{\ell\ge1}(1 - b_\ell) < 1$. We will prove that if $B \subset \{0,1\}^S$ satisfies $\mu(B) > b$, we find $J^1, \ldots, J^q$ in B as in (11.21). We consider the classes B_ℓ constructed from B in Lemma 11.7 and the classes C_ℓ constructed from A in Proposition 11.6. In the first step we use Lemma 11.8 with S_1 instead of S and $A_1, \ldots, A_q = B_1$, $n = 1$ together with the fact that $\mu_1(C_1) \ge 2^{-6}$ to find sets $I^1_1, \ldots, I^q_1 \subset S_1$, $I^1_1, \ldots, I^q_1 \in B_1$ and $X_1 \subset S_1$, $X_1 \in C_1$ with

$$\mathrm{card}(X_1 \setminus (I^1_1 \cup \cdots \cup I^q_1)) \le 1.$$

In the second step we find sets $I^1_2, \ldots, I^q_2 \subset S_2$, $I^\ell_1 \cup I^\ell_2 \in B_\ell$ for $\ell \le q$, and $X_2 \subset S_2$, $X_1 \cup X_2 \in C_2$ with

$$\mathrm{card}(X_2 \setminus (I^1_2 \cup \cdots \cup I^q_2)) \le 2 = 2^{2-1}.$$

This is possible by using Lemma 11.8 with S_2 instead of S and $n = 2$, because conditions (11.31) and (11.39) ensure that there are sufficiently many choices of I^ℓ_2, X^ℓ_2. We then continue in this manner. □

Our next result is a simple argument showing that to prove Conjecture 6.4 it suffices to prove it for arbitrary small values of p. We find it rather symptomatic that we do *not* know how to prove the corresponding result for Conjecture 5.7. This is because it is so difficult to work with p-small classes.

THEOREM 11.9. *Assume that there is a sequence $p_n \to 0$ and a number q with the following property. Given any n, any set S, any class $A \subset \{0,1\}^S$ with $\mu_{p_n}(A) \geq 1 - 1/q$, then $A^{(q)}$ is weakly p_n-small.*

Then for each $0 < p < 1$ and each $A \subset \{0,1\}^S$ with $\mu_p(A) \geq 1 - 1/q$, $A^{(q)}$ is weakly p-small.

Proof. Consider an integer r, and the map $\varphi : (\{0,1\}^S)^r \to \{0,1\}^S$ given by

$$\varphi(J_1, \ldots, J_r) = \bigcup_{\ell \leq r} J_\ell.$$

It is easy to see that if, keeping the dependence in n implicit, we define p' by

$$1 - p' = (1 - p_n)^r,$$

for any class $A \subset \{0,1\}^S$ we have

$$\mu_{p_n}(\varphi^{-1}(A)) = \mu_{p'}(A).$$

Assuming that $\mu_{p'}(A) \geq 1 - 1/q$, and setting $A' = \varphi^{-1}(A)$, we see that by hypothesis $A'^{(q)}$ is weakly p_n-small.

We recall that for two subsets I, Y of S, we define

$$\psi(I,Y) = 1 \text{ if } I \subset Y \text{ and } \psi(I,Y) = 0 \text{ otherwise.}$$

The fact that $A'^{(q)}$ is weakly p_n-small means that there is a probability measure θ' on $(\{0,1\}^S)^r$ such that, whenever $(Y_1, \ldots, Y_r) \in A'^{(q)}$ we have

$$\frac{1}{2} \int \prod_{\ell \leq r} (p_n^{-\mathrm{card} I_\ell} \psi(I_\ell, Y_\ell) d\theta((I_1, \ldots, I_r)) \geq 1. \tag{11.42}$$

Consider $Z \in A^{(q)}$ and

$$W(Z) = \Big\{ (Y_1 \ldots, Y_r) \in (\{0,1\}^S)^r; \quad Z = \bigcup_{\ell \leq r} Y_\ell,\ Y_1, \ldots, Y_r \text{ are disjoint} \Big\}.$$

Thus, if $i \in Z$, and $(Y_1, \ldots, Y_r) \in W(Z)$, there is a unique $\ell \leq r$ with $i \in Y_\ell$. In this manner there is a canonical bijection between $W(Z)$ and $\{0, \ldots, r\}^Z$. The crucial observation is that for $Z \in A^{(q)}$ and any $I_1, \ldots, I_r$ we have

$$\mathrm{Av} \prod_{\ell \leq r} p_n^{-\mathrm{card} I_\ell} \psi(I_\ell, Y_\ell) \leq (p_n r)^{-\mathrm{card} I} \psi(I, Z), \tag{11.43}$$

where $I = \bigcup_{\ell \leq r} I_\ell$, and where Av denotes the average over all elements $(Y_1, \ldots, Y_r)$ of $W(Z)$. This is seen by first observing that the left hand side of (11.43) is zero unless $I_1, \ldots, I_r$ are disjoint, and then that

$$\mathrm{Av} \prod_{\ell \leq r} \psi(I_\ell, Y_\ell) = r^{-\mathrm{card} I} \psi(I, Z).$$

It then follows from (11.42) that if θ is the image measure of θ' under the map $(I_1, \ldots, I_r) \to \bigcup_{\ell \leq r} I_\ell$ then

$$\forall Z \in A^{(q)},\ \frac{1}{2} \int (p_n r)^{-\mathrm{card} I} \psi(I, Z) d\theta(I) \geq 1.$$

Thus $A^{(q)}$ is $p_n r$-small.

Now take $n \to \infty$ and $r = r_n$ such that $p_n r \to -\log(1-p)$, so $p' = 1 - (1 - p_n)^{r_n} \to p$, and we have proved that if $A \subset \{0,1\}^S$ satisfies $\mu_p(A) \geq 1 - 1/q$, then $A^{(q)}$ is $-\log(1-p)$-weakly small and since $p < -\log(1-p)$ it is also p-weakly small. $\square$

We now turn to a general principle that implies roughly that Conjecture 7.8 is true "when there are a lot of symmetries". We assume that ν is the uniform probability on a class B of subsets of S. Each set in B has the same cardinality m. Moreover there is a group Ξ of permutations of S with the following properties:

$$I \in B,\quad \sigma \in \Xi \Rightarrow \sigma(I) \in B. \tag{11.44}$$

$$\begin{aligned} &\text{If } I = \{x_1, \ldots, x_m\} \in B,\ \ J = \{y_1, \ldots, y_m\} \in B, \\ &\qquad \exists \sigma \in \Xi,\ \ \forall i \leq m,\ \ \sigma(x_i) = y_i. \end{aligned} \tag{11.45}$$

To give a non-trivial example, consider the case where S is the collection of all subsets of cardinality k of the set $\{1, \ldots, mk\}$ and where B consists of the partitions of the set $\{1, \ldots, mk\}$ into m sets of k elements. The previous conditions hold when Ξ is the group canonically induced on S by the group of permutations of $\{1, \ldots, mk\}$.

THEOREM 11.10. *There exists a number α with the following property. Whenever the probability ν as above is p-spread, then $\mu = \mu_p$ dominates $W_\alpha(\nu)$.*

Proof. The group Ξ acts on $\{0,1\}^S$. We say that a subset of $\{0,1\}^S$ is Ξ-invariant if it is invariant under this action.

We want to show that

$$W_\alpha(\nu)(A) \leq \mu(A) \tag{11.46}$$

for each up-class A. Since both $W_\alpha(\nu)$ and μ are invariant under the action of Ξ, taking average over this action it suffices to show that

$$\int f dW_\alpha(\nu) \leq \int f d\mu$$

for each monotone Ξ-invariant function f. Considering the classes $\{f \geq a\}$ we see that it suffices to prove (11.46) when A is an up-class Ξ-invariant.

For $k \leq m$ consider the classes

$$B_k = \{J \subset S;\ \mathrm{card} J = k, \exists I \in B,\ J \subset I\},$$

and define

$$k_0 = \inf\{k;\ A \cap B_k\} \neq \varnothing.$$

Then, for $I \in B$ and $J \in A, J \subset I$ we have $\mathrm{card} J \geq k_0$, so that

$$\theta_{\alpha,I}(A) \leq \sum_{k \geq k_0} \alpha^k (1-\alpha)^{m-k} \binom{m}{k},$$

and

$$W_\alpha(\nu)(A) = \int \theta_{\alpha,I}(A) d\nu(I) \leq \sum_{k \geq k_0} \alpha^k (1-\alpha)^{m-k} \binom{m}{k}.$$

Let us define

$$A_k = \{J \subset S;\ \exists I \in B, \mathrm{card}(J \cap I) \geq k\},$$

Then from (11.45) we see that since A is Ξ-invariant we have $A \supset A_{k_0}$.

Thus to prove (11.46) it suffices to prove that

$$b_{k_0} := \sum_{k\geq k_0} \alpha^k (1-\alpha)^{m-k} \binom{m}{k} \leq a_{k_0} := \mu(A_{k_0}). \quad (11.47)$$

It suffices to prove (11.47) when $k_0 \geq 1$, since when $k_0 = 0$ both left and right-hand sides are equal to 1. Also, for $k_0 \geq 1$,

$$b_{k_0} \leq b_1 = 1 - (1-\alpha)^m. \quad (11.48)$$

A first essential observation is that since we assume that ν is p-spread, we have by (11.29) (used for $G_I = H_I$) that

$$a_m = \mu(A_m) \geq 2^{-m}. \quad (11.49)$$

For any m we have $1-(1-\alpha)^m \leq 2^{-m}$ for α small enough (depending on m) so that in this case we have

$$b_{k_0} \leq b_1 = 1-(1-\alpha)^m \leq 2^{-m} \leq a_m \leq a_{k_0},$$

and (11.47) can be obtained for the small values of m simply by taking α small. Thus the issue is the case where m is large enough. We will prove that there is a number L_1 such that

$$L_1 k \leq m \Rightarrow a_k \geq 1 - \exp(-m/L_1). \quad (11.50)$$

First we show that this implies (11.46). The standard estimates on the tail of the binomial law imply that we can assume α small enough that

$$\sum_{m\geq k\geq m/L_1} \alpha^k (1-\alpha)^{m-k} \binom{m}{k} \leq 2^{-m}.$$

Then for $L_1 k_0 \geq m$ we have

$$b_{k_0} \leq 2^{-m} \leq a_m \leq a_{k_0}.$$

On the other hand, if $L_1 k_0 \leq m$ we have

$$b_{k_0} \leq 1-(1-\alpha)^m \leq 1 - \exp\left(-\frac{m}{L_1}\right) \leq a_{k_0}$$

using (11.50) and provided that $1-\alpha \geq \exp(-1/L_1)$. Thus, to complete the proof it remains only to check (11.50). This will follow from (11.29) and Lemma 11.8. Consider the class

$$C = \left\{ I \subset S;\ \forall J \in B,\ \mathrm{card}(I \cap J) < \frac{m}{32} \right\},$$

so that

$$A_m \subset \quad \Big\{ X \subset S;\ \forall I^1, \ldots, I^{16} \in C, \quad \mathrm{card}(X \setminus (I^1 \cup \cdots \cup I^{16})) \geq \frac{m}{2} \Big\}$$

and by (11.49) and (11.40) we have

$$2^{-m} \leq \mu(A_m) \leq \frac{1}{\mu(C)^{16}} \frac{1}{16^{m/2}}$$

and thus $\mu(C) \leq 2^{-m/16}$. Now, $C^c \subset A_k$ for $32k \leq m$, and thus, for $32k \leq m$ we have $a_k \geq 1 - 2^{-m/16}$. □

It would be nice if one could prove that the probability measures of Theorem 11.9 satisfy (9.1). We do not know how to do this, but we at least we know how to handle the following special case.

THEOREM 11.11. *If α is small enough the following occurs. Consider a set S with* card$S = N$, *and consider $k \leq N/2$. Then for each up-class A the uniform probability on the subsets of S of cardinality k satisfies* (9.1) *when $k/N \leq p$.*

We deduce this result from the following.

PROPOSITION 11.12. *If α is small enough the following occurs. Let $p = k/N$, $p' \leq (k-1)/(N-1)$. Denote by μ^i the product measure on $\{0,1\}^N$ such that on the factor of rank i, the weight of 1 is α, and that the weight of 1 is p' on the other factors. Then for each down-class $A \subset \{0,1\}^N$ we have*

$$\mu_p(A) \leq \left(\prod_{i\leq N} \mu^i(A) \right)^{1/N}. \quad (11.51)$$

Proof of Theorem 11.11. Let us denote by μ^{ij} the product measure on $\{0,1\}^N$ such that for the factor of rank i or j, the weight of 1 is α, and is $p'' = (k-2)/(N-2)$ for all the other factors. Then, using Fubini theorem and Proposition 11.12 for $N-1$ and $k-1$ rather than N and k we get for each i that

$$\mu^i(A) \leq \left(\prod_{j\neq i} \mu^{ij}(A) \right)^{1/(N-1)}$$

so that by (11.51)

$$\mu_p(A) \leq \left(\prod_{j\neq i} \mu^{ij}(A) \right)^{1/N(N-1)}$$

and iteration of this process yields (9.2). □

Proof of Proposition 11.12. Let μ_t^i be the product measure on $\{0,1\}^N$ such that on the factor of rank i the weight of 1 is $\alpha(t) = t\alpha + (1-t)p$ while on the other factors it is $p(t) = tp' + (1-t)p$. Let

$$\varphi(t) = \log \prod_{i\leq N} \mu_t^i(A)$$

so that

$$\exp\varphi(1) = \prod_{i\leq N} \mu^i(A) \quad ; \quad \exp\varphi(0) = \mu_p(A)^N$$

and it suffices to prove that $\varphi'(t) \geq 0$. Let $S = \{1, \ldots, N\}$ and for $j \in S$ let

$$A_j = \{X \subset S \setminus \{j\};\quad X \in A,\quad X \cup \{j\} \notin A\}.$$

Let $\mu_{j,t}^i$ be the projection of μ_t^i on $\{0,1\}^{S\setminus\{j\}}$. Then the (totally elementary) Russo-Margulis formula states that

$$\begin{aligned} -\frac{d}{dt}\mu_t^i(A) &= \frac{d}{dt}\alpha(t)\mu_{i,t}^i(A_i) + \frac{d}{dt}p(t)\sum_{j\neq i}\mu_{j,t}^i(A_j) \\ &= (\alpha - p)\mu_{i,t}^i(A_i) + (p'-p)\sum_{j\neq i}\mu_{j,t}^i(A_j) \end{aligned}$$

and thus (exchanging i and j in the double sum)

$$-\varphi'(t) = \sum_{i\leq N}\left((\alpha-p)\frac{\mu_{i,t}^i(A_i)}{\mu_t^i(A)} + (p'-p)\sum_{j\neq i}\frac{\mu_{i,t}^j(A_i)}{\mu_t^j(A)} \right). \quad (11.52)$$

We have to show that the right-hand side is ≤ 0. The basic reason is that

$$p' - p = \frac{k-1}{N-1} - \frac{k}{N} = -\frac{N-k}{N(N-1)} \leq -\frac{1}{2(N-1)}$$

so that $N-1$ terms with this coefficient easily outweigh a term with coefficient $\alpha - p \leq \alpha$. To check the details, let us fix $i \neq j$, let

$$B = \{X \subset S \setminus \{i,j\}; \quad X \in A\}$$

and let μ_t be the projection of μ_t^i on $\{0,1\}^{S \setminus \{i,j\}}$, which is also the projection of μ_t^j on the same space. (It is the product measure that gives weight $p(t)$ to 1 in each factor). Then, since A is a down-class, we have

$$\mu_t^i(A) \leq \mu_t(B)$$

and also

$$\mu_t^i(A) \geq (1-\alpha(t))(1-p(t))\mu_t(B) \geq \frac{1}{4}\mu_t(B).$$

The same inequalities hold for $\mu_t^j(A)$, and thus $\mu_t^j(A) \leq 4\mu_t^i(A)$. Thus, to show that the right-hand side of (11.52) is ≤ 0 it suffices to show that for each i we have

$$4\alpha\mu_{i,t}^i(A_i) \leq \frac{1}{2(N-1)} \sum_{j \neq i} \mu_{i,t}^j(A_i). \qquad (11.53)$$

If we make explicit the contribution of the factor of rank j, we see that for certain numbers $a_j \geq b_j$ we have

$$\begin{aligned} \mu_{i,t}^j(A_i) &= \alpha(t)a_j + (1-\alpha(t))b_j \\ \mu_{i,t}^i(A_i) &= p(t)a_j + (1-p(t))b_j \end{aligned}$$

and since $1/2 \geq \alpha(t) \geq p(t)$ we have $\mu_{i,t}^j(A_i) \geq \mu_{i,t}^i(A_i)$. Thus (11.53) holds if $\alpha = 1/8$. □

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