Matchings

Consider two independent sequences $(X_i)_{i\le N}$, $(Y_i)_{i\le N}$ i.i.d. uniform in the unit square $[0,1]^2$. We denote by X_i^1 and X_i^2 the components of X_i, and same for Y_i. A matching is simply a permutation π of $\{0,\dots,N\}$.

Problem (The ultimate matching conjecture). Prove or disprove the following. We can find a univeral constant L with the following property. Consider $\alpha_1,\alpha_2 > 0$ with $1/\alpha_1 + 1/\alpha_2 = 1/2$. Then with high probability $\ge 1/2$ we can find a matching π such that, for $j=1,2$, we have

$$\sum_{i\le N} \exp\left(\sqrt{\frac{N}{\log N}}\frac{|X_i^j - Y_{\pi(i)}^j|}{L}\right)^{\alpha_j} \le 2N\ . \tag{1}$$

A particulary interesting case is $\alpha_1=\infty$ and $\alpha_2=2$. In that case for $j=1$ (1) means that

$$\max_{i\le N}|X_i^1 - Y_{\pi(i)}^1| \le L\frac{\sqrt{\log N}}{\sqrt{N}}\ . \tag{2}$$

In that case the best I can do is to prove (1) for $j=2$ for any $\alpha_2 < 1/2$ (with a constant $L(\alpha_2)$).

Another interesting case is $\alpha_1=\alpha_2=4$. Noting that

$$\sum_{i\le N}\exp a_i^4 \le 2N \Rightarrow \max_{i\le N}|a_i| \le L(\log N)^{1/4}\,,$$

and denoting by d the distance in the plane then (1) implies

$$\max_{i\le N} d(X_i, Y_{\pi(i)}) \le L\frac{(\log N)^{3/4}}{\sqrt{N}}\ . \tag{3}$$

and of course also

$$\sum_{i\le N} d(X_i, Y_{\pi(i)}) \le L\sqrt{N}\sqrt{\log N}\ . \tag{4}$$

The existence of a matching satisfying (3) is due to Leighton and Shor, and of a matching satisfying (4) is the famous AKT theorem. See "upper and lower bounds..." for proofs. I can't even prove that there is a matching satisfying both (3) and (4). You get the \$ 1000 prize even if you do not solve the general problem but one of the two special cases above.