An assignment problem at high temperature

by Michel Talagrand

Abstract.

Physicists have studied the stochastic assignment problem using ideas from statistical mechanics. For a version of this problem, we give, at high enough temperature, a complete proof of the existence of the structure they predict.

I. Introduction.

Given positive numbers $(a_{i,j})_{i,j \le N}$, the assignment problem is to find

$$\min_{\sigma} \sum_{i \le N} a_{i,\sigma(i)} \tag{1.1}$$

where σ ranges over all permutations of $\{1, ..., N\}$. In words, $a_{i,j}$ is the cost of assigning job j to worker i. We are required to assign exactly one job to each worker, and we try to do this at the minimum possible cost. In this paper, we are interested in the stochastic version of the problem, where the numbers $a_{i,j}$ are independent uniformly distributed over $[0,1]$.

Physicists have studied this model using ideas from statistical mechanics. The basis of their approach is to introduce the Hamiltonian

$$H_N(\sigma) = N \sum_{i \le N} a_{i,\sigma(i)}, \tag{1.2}$$

and an "inverse temperature" λ , and to provide the set of all the permutations on N with Gibbs' measure, a probability measure of density proportional to $\exp(-\lambda H_N(\sigma))$. They use the (non rigorous) methods they have developed for the study of “spin glasses”, and this paper is part of the author’s program to rigorously prove (at least at high temperature) what the physicist discovered.

The most famous prediction of the physicists [M-P1, M-P2, M-P-V] is that, for large N, the quantity (1.1) is asymptotically $\pi^2/6$. After this paper was submitted, D. Aldous [A2] did give a rigorous (and remarkable) proof of this result, following a very different route and taking advantage of a special feature, namely the existence of a “ limiting object ” as $N \to \infty$ [A1]. This is however not the whole story, and, as a desordered system, the present model remains of interest, in particular because it seems to differ considerably from the other models (Sherrington-Kirkpatrick, Hopfield, K-sat, perceptron capacity) that the author previously investigated. Being familiar with this previous work would probably provide no help to penetrate the present paper, and the present discussion is provided only for comparison purposes. The physicists predict that for *all* values of λ the model is in a “high temperature” phase, or, as they say, that the replica-symmetric solution holds. Proving that this is the case is a problem of the type “ controling the entire high temperature region” and currently these problems are very difficult. What one would like to prove in a first stage is that the physicists predictions are correct if $\lambda \le \lambda_0$, where λ_0 is a given number. This is a “very high temperature” hypothesis, and the author could do this in the four previously mentioned models under such an hypothesis. Unfortunately, we could not completely reach this goal in the present case. One reason is probably that this model is very different from the models previously considered. In fact, despite previous results on the other models, the author stared at the present model for quite a while without being able to say anything at all. The techniques we will use are related to the techniques used on the four previously mentionned models only at a high level. This is why this paper can be read without any prior knowledge.

Not being able to prove what we want, we will study a slightly different model. This model will also exhibit a structure similar to that predicted for the original model.

We consider an integer $M \ge N$, and

$$H_{N,M}(\sigma) = N \sum_{i \le N} a_{i,\sigma(i)}, \tag{1.3}$$

where now σ is a one to one map from $\{1, ..., N\}$ to $\{1, ..., M\}$. The r.v. $(a_{i,j})_{i \le N, j \le M}$ are of course i.i.d. uniform. We will take $N \to \infty, M \to \infty$, with a “fixed ratio”, that is

$M = \lfloor (1+\alpha)N \rfloor$, where $\alpha > 0$.

It is probably not apparent now to the reader why this model is easier than the case $M = N$. This should become clear in Section 2. We will obtain a good description of the model when $\lambda \le \lambda_0(\alpha)$. It is very likely that the picture we provide is also true for $\lambda \le \lambda_0$ (where now λ_0 does not depend upon α), but proving this should be very much like considering the case $N = M$. (As explained, we do not know how to do this.)

Let us now describe, informally first, some of what we can prove. It is natural to consider the following quantity ("partition function"), which is the normalizing factor in Gibbs' measure

$$S_{N,M} = \sum \exp(-\lambda H_{N,M}(\sigma)), \tag{1.4}$$

where the summation is of course over all one to one maps σ from $\{1, ..., N\}$ to $\{1, ..., M\}$. Given $i \le N, j \le M$ we will also consider

$$S_{N,M}(i;j) = \sum \exp(-\lambda N \sum_{k \ne i} a_{k,\sigma(k)}), \tag{1.5}$$

where the summation is now over all one to one maps σ from $\{1, ..., N\}\backslash\{i\}$ to $\{1, ..., M\}\backslash\{j\}$. This quantity is closely related to Gibbs' measure. If we denote by $G_{N,M}$ this Gibbs' measure, then

$$G_{N,M}(\{\sigma(i) = j\}) = \exp(-\lambda N a_{i,j}) \frac{S_{N,M}(i;j)}{S_{N,M}}.$$

One of the basic intuitions provided by physics is that (in the high temperature region) "the spins are nearly independent under Gibbs' measure". This means here that given $i_1 \ne i_2$, the laws under Gibbs' measure of the variables $\sigma \to \sigma(i_1)$ and $\sigma \to \sigma(i_2)$ are nearly independent. It is however not easy to find a convenient quantitative version of this statement, and instead we will work with the quantities $S_{N,M}(i;j)$.

Central to our approach is the fact that the $N \times M$ random matrix $(S_{N,M}(i,j))_{i \le N, j \le M}$. "decomposes"

$$\frac{S_{N,M}(i;j)}{S_{N,M}} \simeq z_{N,M}(i) u_{N,M}(j) \tag{1.6}$$

for certain random quantities $z_{N,M}(i), u_{N,M}(j)$. Moreover, these quantities are of the same nature as (1.5). Namely,

$$z_{N,M}(i) = \frac{S_{N,M}(i;\emptyset)}{S_{N,M}}; u_{N,M}(j) = \frac{S_{N,M}(\emptyset;j)}{S_{N,M}}. \tag{1.7}$$

There,

$$S_{N,M}(i;\emptyset) = \sum \exp(-\lambda N \sum_{k \neq i} a_{k,\sigma(k)}), \tag{1.8}$$

where the summation is over all one to one maps σ from $\{1, ..., N\} \setminus \{i\}$ to $\{1, ..., M\}$, and

$$S_{N,M}(\emptyset;j) = \sum \exp(-\lambda H_{N,M}(\sigma)), \tag{1.9}$$

where the summation is now over all maps σ from $\{1, ..., N\}$ to $\{1, ..., M\} \setminus \{j\}$.

Clearly (1.6) is related to the emergence of independence properties. This relation might be more intutive if we rewrite it as

$$\frac{S_{N,M}(i;j)}{S_{N,M}(i;\emptyset)} = \frac{S_{M,N}(\emptyset;j)}{S_{M;N}(\emptyset;\emptyset)}.$$

The right-hand side is the Gibbs' probability that j does not belong to the range of σ. The left-hand side is the Gibbs' probability of the same event when one has removed i from $\{1, \cdots, N\}$ and one has replaced λ by λ' such that $\lambda'(N-1) = \lambda N$.

A way to express (1.6) is that the relative variation of $S_{N,M}$ when we remove i and j from their respective index sets is nearly the product of the relative variations when one removes only one of these indices. This fact can be generalized to any given number of indices.

What about the quantities $u_{N,M}, z_{N,M}$? We will show that there exists two probability measures μ_u, μ_z (depending upon λ, α only) such that

$$\frac{1}{M} \sum_{j \leq M} \delta_{u_{N,M}(j)} \sim \mu_u. \tag{1.10}$$

$$\frac{1}{N} \sum_{i \leq N} \delta_{z_{N,M}(i)} \sim \mu_z. \tag{1.11}$$

(In particular, the left-hand sides are essentially non random). The probabilities μ_u, μ_z will be described as fixed points of certain operators.

Even though this might not be apparent yet, once the previous results have been obtained, all kinds of questions can be answered, for example the computation of the "free energy".

We now state our results formally

Theorem 1.1. *Given $\alpha > 0$, there is a number $\lambda_0(\alpha) > 0$ such that if $N \to \infty; M = \lfloor N(1+\alpha) \rfloor, \lambda \le \lambda_0(\alpha)$, for all $i \le N, j \le M$ we have*

$$E((\frac{S_{N,M}(i,j)}{S_{N,M}} - z_{N,M}(i)u_{N,M}(j))^2) \le \frac{K(\alpha)}{N}, \tag{1.12}$$

where $K(\alpha)$ depends upon α only. Finaly, there exists two probability measures μ_u, μ_z such that

$$\lim_{N\to\infty} E(\Delta(\frac{1}{N}\sum_{i\le N} \delta_{z_{N,M}(i)}, \mu_z)) = 0 \tag{1.13}$$

$$\lim_{N\to\infty} E(\Delta(\frac{1}{M}\sum_{j\le M} \delta_{u_{N,M}(j)}, \mu_u)) = 0, \tag{1.14}$$

where Δ denotes the square of Wasserstein's distance. Moreover,

$$\lim_{N\to\infty} E\frac{1}{N}\log S_{N,M} = -\int \log x d\mu_u(x) - (1+\alpha)\int \log x d\mu_z(x). \tag{1.15}$$

We hope that this theorem makes the interest and the beauty of this model apparent. It seems to be always the case that once one has succeeded to compute the quantity (1.15), one has also developed enough tools to have a good hold on the model, and one can describe it in detail. This seems to be also the case here. We will not attempt to do this, but to illustrate this fact, we will sketch how to describe some features of Gibbs' measure.

The next natural step should be to prove Theorem 1.1 under the condition $\lambda \le \lambda_0$ rather than $\lambda \le \lambda_0(\alpha)$, but the real goal is to control the model for all values of λ. We think that this is a very interesting question. The cavity method that we develop here seems to be currently a necessary ingredient to such a result, but it does not seem (as in all the other models) that by itself it can lead to a control of the entire range of λ. One approach worth trying would be to combine the cavity method with a priory estimates for the Gibbs' measure, a strategy that works well in the case of the Hopfield model. David Aldous proved [A1] that the optimal assignment is unique in the strong sense that near-optimal assignments must be close to it. A quantitatively strong enough version of this result might provide the required estimates, but this is better left for future research.

The proof of (1.12) will occupy much of Sections 2 and 3. The proof of (1.13) and (1.14) will occupy much of Sections 4 and 5, and is rather technical. Probably the reader should first look at the simplest argument, the proof of (1.15), towards the end of the paper. It provides motivation for some of the previous considerations.

2. Starting the cavity method.

Throughout the paper, we will write

$$c_{i,j} = \exp(-\lambda N a_{i,j}), \tag{2.1}$$

so that, in particular, $c_{i,j} \le 1$.

Given a subset A of $\{1, ..., N\}$, a subset B of $\{1, ..., M\}$, with

$$N - \mathrm{card} A \le M - \mathrm{card} B,$$

we write

$$S_{N,M}(A; B) = \sum \prod c_{i,\sigma(i)}, \tag{2.2}$$

where the product is over $i \in \{1, ..., N\} \setminus A$, and the summation over all one to one maps from $\{1, ..., N\} \setminus A$ to $\{1, ..., M\} \setminus B$. If $A = \{i_1, i_2, ...\}$, $B = \{j_1, j_2, ...\}$, we write

$$S_{N,M}(A; B) = S_{N,M}(i_1, i_2, ...; j_1, j_2, ...).$$

This is consistent with the notation (1.4), (1.5), (1.8), (1.9) (with $S_{N,M} = S_{N,M}(\emptyset; \emptyset)$). The following will be fundamental

Lemma 2.1. *If $i \notin A$ we have*

$$S_{N,M}(A; B) = \sum_{\ell \notin B} S_{N,M}(A \cup \{i\}; B \cup \{\ell\}) c_{i,\ell}. \tag{2.3}$$

If $j \notin B$, we have

$$S_{N,M}(A; B) = S_{N,M}(A; B \cup \{j\}) + \sum_{k \notin A} S_{N,M}(A \cup \{k\}; B \cup \{j\}) c_{k,j}. \tag{2.4}$$

Proof. The proof consists in replacing $S_{N,M}(.;.)$ by its value and checking that indeed the same terms occur in the left-hand side and the right-hand side. Any further comment is more likely to be a hindrance than a help. ∎

Lemma 2.2. *If $M \notin B$ we have*

$$S_{N,M}(A; B \cup \{M\}) = S_{N,M-1}(A; B). \tag{2.5}$$

If $N \notin A$, we have

$$S_{N,M}(A \cup \{N\}; B) = S_{N-1,M}(A; B). \tag{2.6}$$

It is understood in (2.6) (and in similar situations below) that in the system relative to $N-1, M$, the value of λ has been slightly changed, into a value λ' such that $\lambda'(N-1) = \lambda N$.

Proof. These are again obvious identities . $\square$

We recall the definitions of Section 1 :

$$u_{N,M}(j) = \frac{S_{N,M}(\emptyset; j)}{S_{N,M}(\emptyset; \emptyset)}; \tag{2.7}$$

$$z_{N,M}(i) = \frac{S_{N,M}(i; \emptyset)}{S_{N,M}(\emptyset; \emptyset)}. \tag{2.8}$$

There two quantities are closely related, as the following shows.

Lemma 2.3. *We have*

$$u_{N,M}(M) = \frac{1}{1 + \sum_{k \le N} z_{N,M-1}(k) c_{k,M}} \tag{2.9}$$

$$z_{N,M}(N) = \frac{1}{\sum_{\ell \le M} u_{N-1,M}(\ell) c_{N,\ell}}. \tag{2.10}$$

Proof. To prove (2.9), we use (2.4) with $A = B = \emptyset$ in the denominator of (2.7), with $j = M$. We then use (2.5), with $A = \{k\}, B = \emptyset$. To prove (2.8), we proceed similarly, using now (2.9) and (2.6). $\square$

We now consider

$$A_{N,M}(i,j) = \frac{S_{N,M}(i; \emptyset) S_{N,M}(\emptyset; j) - S_{N,M}(\emptyset; \emptyset) S_{N,M}(i; j)}{S_{N,M}(\emptyset; \emptyset)^2}. \tag{2.11}$$

The motivation is simply that the left-hand side of (1.12) is $EA_{N,M}(i,j)$ (which is independent of i, j). Together with (2.11), we consider the following quantity, of a similar nature

$$R_{N,M}(j,\ell) = \frac{S_{N,M}(\emptyset; \emptyset) S_{N,M}(\emptyset; j, \ell) - S_{N,M}(\emptyset; j) S_{N,M}(\emptyset; \ell)}{S_{N,M}(\emptyset; \emptyset)^2}. \tag{2.12}$$

The basis of the method of proof of (1.12) is to relate $EA^2_{N,M}(i,j)$ with $ER^2_{N-1,M}(j,\ell)$ and $ER^2_{N,M}(j,\ell)$ with $EA^2_{N,M-1}(i,j)$. We will then obtain (1.12) through iteration of these relations. The "algebraic" part of the proof is the following lemma, that is of a nature similar to Lemma 2.3 (but more complicated).

Lemma 2.4. *We have*

$$R^2_{N,M}(M,j)=\frac{(\sum_{k\le N} c_{k,M}A_{N,M-1}(k,j))^2}{(1+\sum_{k\le N} c_{k,M}z_{N,M-1}(k))^4} \tag{2.13}$$

$$A^2_{N,M}(N,j)=\frac{(\sum_{\ell\le M,\ell\ne j} c_{N,\ell}R_{N-1,M}(\ell,j)-c_{N,j}u^2_{N-1,M}(j))^2}{(\sum_{\ell\le M} c_{N,\ell}u_{N-1,M}(\ell))^4} \tag{2.14}$$

Proof. This is again a consequence of Lemmas 2.1, 2.2. To obtain (2.13), we simply use in (2.12) the relations

$$\begin{aligned}
S_{N,M}(\emptyset;\emptyset)&=S_{N,M-1}(\emptyset;\emptyset)+\sum_{k\le N} c_{k,M}S_{N,M-1}(k;\emptyset)\\
S_{N,M}(\emptyset;j,M)&=S_{N,M-1}(\emptyset;j)\\
S_{N,M}(\emptyset;j)&=S_{N,M-1}(\emptyset;j)+\sum_{k\le N} c_{k,M}S_{N,M-1}(k;j)\\
S_{N,M}(\emptyset;M)&=S_{N,M-1}(\emptyset;\emptyset)
\end{aligned}.$$

(which follow from Lemmas 2.1, 2.2) and we regroup the terms in the numerator. To prove (2.14), we substitute in (2.11) the relations

$$\begin{aligned}
S_{N,M}(N;\emptyset)&=S_{N-1,M}(\emptyset;\emptyset)\\
S_{N,M}(\emptyset;j)&=\sum_{\ell\ne j} S_{N-1,M}(\emptyset;\ell,j)c_{N,\ell}\\
S_{N,M}(\emptyset;\emptyset)&=\sum_{\ell\le N} S_{N-1,M}(\emptyset;\ell)c_{N,\ell}\\
S_{N,M}(N;j)&=S_{N-1,M}(\emptyset;j)
\end{aligned}$$

and we regroup the terms. ∎

In the use of (2.14), the following will be essential (as will become apparent in Section 3).

Lemma 2.5. *We have*

$$\sum_{\ell\le M} u_{N,M}(\ell)=M-N. \tag{2.15}$$

Proof. This means that

$$\sum_{\ell\le M} S_{N,M}(\emptyset;\ell)=(M-N)S_{N,M}(\emptyset;\emptyset). \tag{2.16}$$

This is true because each term of $S(\emptyset;\emptyset)$ appears $M-N$ times on the right. ∎

Lemma 2.6. *We have*

$$\sum_{\ell\le M,\ell\neq j} R_{N,M}(\ell,j) = -u_{N,M}(j)+u_{N,M}^2(j). \tag{2.17}$$

Proof. Using (2.16), we have

$$\sum_{\ell\le M,\ell\neq j} S_{N,M}(\emptyset;\ell) = (M-N)S_{N,M}(\emptyset;\emptyset) - S_{N,M}(\emptyset;j)$$

Moreover, we have the following (similar to (2.16))

$$\sum_{\ell\le M,\ell\neq j} S_{N,M}(\emptyset;\ell,j) = (N-M-1)S_{N,M}(\emptyset;j).$$

The result follows in a straightforward manner. □

3. Decoupling.

In this section we prove (1.12). A basic observation is that in the right-hand side of (2.13), the quantities $c_{k,M}$ are probabilistically independent of the quantities $A_{N,M-1}(k)$ and $z_{N,M-1}(k)$. Thus, when taking expectation, we can first take expectation in $c_{k,M}$ at $A_{N,M-1}(k), z_{N,M-1}(k)$ given.

Rather than (2.13), we will use the following

$$R_{N,M}^2(M,j) \le (\sum_{k\le N} c_{k,M}A_{N,M-1}(k,j))^2. \tag{3.1}$$

While doing this, we gain two things. First, we do not have to know anything about the numbers $z_{N,M-1}(k)$. Second, the expectation will be much easier to take. On the other hand, this bound is very crude. (It will turn out later that $\sum_{k\le N} c_{k,M}z_{N,M-1}(k)$ is typically of order $1/\alpha$, so that in (3.1) we lose a factor α^4).

We now consider an i.i.d sequence (X_k), uniform on $[0,1]$, and we set $c_k = \exp(-\lambda N X_k)$. The following is obvious.

Lemma 3.1. *We have*

$$Ec_k^p = \frac{1}{\lambda pN}(1-e^{-\lambda pN}) \le \frac{1}{\lambda pN}. \tag{3.2}$$

Lemma 3.2. *For numbers (a_k), we have*

$$E(\sum_{k\leq N} c_k a_k)^2 \leq (\frac{1}{\lambda^2 N}+\frac{1}{2\lambda N})(\sum_{k\leq N} a_k^2). \tag{3.3}$$

Proof. We write

$$EX^2=(EX)^2+E(X-EX)^2$$

and we use (3.2) for $p=1,2$, so that

$$\begin{aligned} E(\sum_{k\leq N} c_k a_k)^2 &\leq (\frac{1}{\lambda N})^2(\sum_{k\leq N} a_k)^2+\frac{1}{2\lambda N}(\sum_{k\leq N} a_k^2) \\ &\leq (\frac{1}{\lambda^2 N}+\frac{1}{2\lambda N})(\sum_{k\leq N} a_k^2) \end{aligned}$$

using Cauchy-Schwarz. □

If we combine with (3.1), (and use that $EA^2_{N,M-1}(k,j)=EA^2_{N,M-1}(k',j)$ for all k,k') we obtain that

Corollary 3.3. *If $\lambda\leq 1$, we have, for any $k\leq N$*

$$ER^2_{N,M}(M,j)\leq \frac{2}{\lambda^2}EA^2_{N,M-1}(k,j). \tag{3.4}$$

Our next goal is to use (2.14) to complement (3.4). This requires more work. We denote by K a universal constant (independent of λ, N, etc) that need not be the same at each occurrence.

Lemma 3.4. *Consider an integer S. Then*

$$S\geq 20\lambda N \Rightarrow E\frac{1}{(\sum_{k\leq S} c_k)^8}\leq K(\frac{\lambda N}{S})^8. \tag{3.5}$$

Proof. If $X=\sum_{k\leq S} c_k$, we write, for $\mu>0$

$$\begin{aligned} P(X\leq t) &\leq e^{\mu t}E\exp(-\mu X) \\ &= e^{\mu t}\Pi_{k\leq S}E\exp(-\mu c_k). \end{aligned} \tag{3.6}$$

We observe that

$$E\exp(-\mu c_k)\leq 1-\frac{1}{2}P(\mu c_k\geq 1),$$

and, for $x < 1, x > e^{-\lambda N}$,

$$P(c_k \geq x) = \frac{1}{\lambda N} \log \frac{1}{x}, \tag{3.7}$$

so that, if $\mu \geq 1, \mu \leq e^{\lambda N}$,

$$\begin{aligned} E \exp(-\mu c_k) &\leq 1 - \frac{1}{2\lambda N} \log \mu \\ &\leq \exp(-\frac{1}{2\lambda N} \log \mu), \end{aligned}$$

and, by (3.6),

$$P(X \leq t) \leq e^{\mu t} (\frac{1}{\mu})^{\frac{S}{2\lambda N}}.$$

Taking $t = S/(\lambda N \mu)$ we get, for $\mu \geq 1, \mu \leq e^{\lambda N}$

$$P(\frac{1}{X} \geq \frac{\lambda N}{S} \mu) = P(X \leq \frac{S}{N \lambda \mu}) \leq (\frac{e^2}{\mu})^{\frac{S}{2\lambda N}}, \tag{3.8}$$

from which the result follows, using that $X \geq S e^{-\lambda N}$.

Proposition 3.5. *Consider numbers* $(w_\ell)_{\ell \leq M}, (u_\ell)_{\ell \leq M}, (u'_\ell)_{\ell \leq M}$, *and* $b > 0$. *We assume that*

$$0 \leq u_\ell, u'_\ell \leq 1, \sum_{\ell \leq M} u_\ell = \sum_{\ell \leq M} u'_\ell = Mb \tag{3.9}$$

$$\lambda \leq \frac{b}{80}. \tag{3.10}$$

Then, if we set $\dot{c}_\ell = c_\ell - Ec_\ell$, *we have*

$$E \left(\frac{\sum_{\ell \leq M} \dot{c}_\ell w_\ell}{(\sum_{\ell \leq M} c_\ell u_\ell)(\sum_{\ell \leq M} c_\ell u'_\ell)} \right)^2 \leq \frac{K \lambda^3}{b^8} \left(\frac{1}{N} \sum_{\ell \leq M} w_\ell^2 \right). \tag{3.11}$$

Comment. A crucial fact is that we have a factor λ^3 (rather than λ^2) on the right. It is essential for this to have $\dot{c}_\ell$ rather than c_ℓ in the numerator of (3.11).

Proof. The temptation is to use Lemma 3.4 and Hölder's inequality. This does not work, because when computing $E(\sum_{\ell \leq M} \dot{c}_\ell w_\ell)^4$, we get a term containing $\sum_{\ell \leq M} w_\ell^4$, and we do not know that this is about $N^{-1}(\sum_{\ell \leq M} w_\ell^2)^2$. To go around this difficulty, we introduce a parameter L, and the set

$$I = \{\ell \leq M; w_\ell^2 \geq \frac{L}{M} \sum_{k \leq M} w_k^2\}. \tag{3.12}$$

Thus,

$$\sum_{\ell\in I} w_\ell^2 \geq \frac{L\mathrm{card} I}{M}\sum_{k\leq M} w_k^2,$$

and thus

$$\mathrm{card} I \leq \frac{M}{L}. \tag{3.13}$$

Next, we consider the set

$$J_1 = \{\ell \leq M; u_\ell \geq \frac{b}{2}\},$$

so that

$$\sum_{\ell\in J_1} u_\ell \geq \frac{bM}{2},$$

and since $u_\ell \leq 1$, we have $\mathrm{card} J_1 \geq bM/2$.

Thus, if we choose

$$L = \frac{4}{b} \tag{3.14}$$

we have

$$\mathrm{card} J \geq \frac{bM}{4} \tag{3.15}$$

where

$$J = \{\ell \leq M, \ell \notin I, u_\ell \geq \frac{b}{2}\}.$$

We define

$$J' = \{\ell \leq M; \ell \notin I, u'_\ell \geq \frac{b}{2}\},$$

and we also have

$$\mathrm{card} J' \geq \frac{bM}{4}. \tag{3.16}$$

Now, the left-hand side of (3.10) is at most

$$2(U+V),$$

where

$$U = E\frac{(\sum_{\ell\in I}\dot{c}_\ell w_\ell)^2}{(\sum_{\ell\in J} c_\ell u_\ell)^2(\sum_{\ell\in J'} c_\ell u'_\ell)^2} \tag{3.17}$$

$$V = E\frac{(\sum_{\ell \notin I} \dot{c}_\ell w_\ell)^2}{(\sum_{\ell \in J} c_\ell u_\ell)^2 (\sum_{\ell \in J'} c_\ell u'_\ell)^2}. \tag{3.18}$$

The study of U is made much easier by the fact that $I \cap (J \cup J') = \emptyset$, so that the numerator and the denominator are independent. Thus (using Hölder's inequality)

$$U \le E(\sum_{\ell \in I} \dot{c}_\ell w_\ell)^2 E\left(\frac{1}{(\sum_{\ell \in J} c_\ell u_\ell)^8}\right)^{1/4} E\left(\frac{1}{(\sum_{\ell \in J'} c_\ell u'_\ell)^8}\right)^{1/4}. \tag{3.19}$$

We appeal to Lemma 3.4 to see using (3.15) that

$$\lambda \le \frac{b}{80} \Rightarrow \left(E\left(\frac{1}{\sum_{\ell \in J} c_\ell u_\ell}\right)^8\right)^{1/4} \le \frac{K\lambda^2}{b^4}, \tag{3.20}$$

and similarly for J'. Thus, since $E\dot{c}_\ell^2 \le 1/2\lambda N$, we get, under (3.10) that

$$U \le \frac{K\lambda^3}{b^8}(\frac{1}{N}\sum_{\ell \le M} w_\ell^2). \tag{3.21}$$

To study V, we use Hölder's inequality to get

$$\begin{aligned} V &\le (E(\sum_{\ell \notin I} \dot{c}_\ell w_\ell)^4)^{1/2} E(\frac{1}{(\sum_{\ell \in J} c_\ell u_\ell)^8})^{1/8} E(\frac{1}{(\sum_{\ell \in J'} c_\ell u'_\ell)^8})^{1/8} \\ &\le \frac{K\lambda^4}{b^8}(E(\sum_{\ell \notin I} \dot{c}_\ell w_\ell)^4)^{1/2} \end{aligned} \tag{3.22}$$

under (3.10), using (3.20) again.

Now,

$$E(\sum_{\ell \notin I} \dot{c}_\ell w_\ell)^4 \le K(\sum_{\ell \notin I} E\dot{c}_\ell^4 w_\ell^4 + \sum_{k,\ell} E\dot{c}_\ell^2 E\dot{c}_k^2 w_k^2 w_\ell^2). \tag{3.23}$$

Since $E\dot{c}_\ell^4 \le K/\lambda N, E\dot{c}_\ell^2 \le K/\lambda N$, and since for $\ell \notin I$ we have (by definition of I)

$$w_\ell^4 \le w_\ell^2(\frac{L}{M}\sum_{k \le M} w_k^2),$$

we get that

$$E(\sum_{\ell \notin I} \dot{c}_\ell w_\ell)^4 \le K(\frac{1}{\lambda^2} + \frac{L}{\lambda})(\frac{1}{N}\sum_{k \le M} w_k^2)^2. \tag{3.24}$$

If we recall (3.10), (3.14), we get that

$$\left(E(\sum_{\ell \in I} \dot{c}_\ell w_\ell)^4\right)^{1/2} \le \frac{K}{\lambda}(\frac{1}{N}\sum_{k \le M} w_k^2),$$

and this completes the proof. □

Corollary 3.6. *If $b = (M-N)/M$ then under (3.10) we have, for any $\ell \le N-1, \ell \ne j$*

$$EA^2_{N,M}(N,j) \le \frac{K\lambda^3}{b^8}\Big(\frac{1}{1-b}ER^2_{N-1,M}(\ell,j) + \frac{1}{N}\Big). \tag{3.25}$$

Proof. Using (2.17), we see that for j fixed,

$$\sum_{\ell \le M, \ell \ne j} c_{N,\ell}R_{N-1,M}(\ell,j) - c_{N,j}u^2_{N-1,M}(j) = \sum_{\ell \le M} \dot{c}_{N,\ell}w_\ell + u_{N,M}(j)Ec_{N,j},$$

where $w_\ell = R_{N-1,M}(\ell,j)$ if $\ell \ne j$ and $w_j = -u^2_{N-1,M}(j)$, and where $\dot{c}_{N,\ell} = c_{N,\ell} - Ec_{N,\ell}$. Thus, by (2.14), and since $u_{N,M}(j) \le 1$,

$$EA^2_{N,M}(N,j) \le 2E\frac{(\sum_{\ell \le M} \dot{c}_{N,\ell}w_\ell)^2 + \frac{1}{\lambda N}}{(\sum_{\ell \le M} c_{N,\ell}u_{N-1,M}(\ell))^4}. \tag{3.26}$$

Setting $u_\ell = u'_\ell = u_{N-1,M}(\ell)$, we see from (2.15) that (3.9) holds for $b = (M-N)/M$.

In the right-hand side of (3.26) we take expectation in $c_{\ell,N}$ first. We use (3.11), and the fact that

$$Ew^2_\ell = ER^2_{N-1,M}(\ell,j) \quad \text{if} \quad \ell \ne j$$

$$Ew^2_j \le 1;$$

We then observe that $M/N = 1/(1-b)$. □

Proof of (1.12). If we combine (3.25) and (3.4), we get,

$$ER^2_{N,M}(N,j) \le \frac{K\lambda}{b^8 N} + \frac{K\lambda}{b^8(1-b)}ER^2_{N-1,M-1}(k,\ell), \tag{3.27}$$

where $b = (M-1-N)/(M-1)$.

In particular, if $M \ge N(1+\alpha/2)$ (and, say, $M \le 2N$ to avoid trivial complications), we get

$$\lambda \le \frac{\alpha^8}{K} \Rightarrow ER^2_{N,M}(N,j) \le \frac{K\lambda^3}{b^8 N} + \frac{1}{2}ER^2_{N-1,M-1}(k,\ell)$$

from which (3.8) follows by iteration since $R^2_{N,M} \le 4$. □

Besides (1.12), there exist similar relations that will be useful for the sequel.

Proposition 3.7. *Under the conditions of Theorem 1.1, we have, for $j \le M-1, i \le N-1$,*

$$E((u_{N,M}(j) - u_{N,M-1}(j))^2) \le \frac{K(\alpha)}{N} \tag{3.28}$$

$$E((u_{N,M}(j) - u_{N-1,M}(j))^2) \le \frac{K(\alpha)}{N} \tag{3.29}$$

(3.30)
$$E((z_{N,M}(i) - z_{N,M-1}(i))^2) \le \frac{K(\alpha)}{N}$$

(3.31)
$$E((z_{N,M}(i) - z_{N-1,M}(i))^2) \le \frac{K(\alpha)}{N}.$$

Proof. The proofs of these are similar, so let us prove only (3.29). We have

$$u_{N,M}(j) - u_{N-1,M}(j) = \frac{S_{N,M}(\emptyset;j)S_{N,M}(N;\emptyset) - S_{N,M}(\emptyset;\emptyset)S_{N,M}(N;j)}{S_{N,M}(\emptyset;\emptyset)S_{N,M}(N;\emptyset)}$$

and, proceeding as in the proof of Lemma 2.4, this is

$$\frac{\sum_{\ell\le M,\ell\ne j} c_{N,\ell}R_{N-1,M}(\ell,j) - c_{N,j}u^2_{N-1,M}(j)}{\sum_{\ell\le M} u_{N-1,M}(\ell)c_{N,\ell}}.$$

Proceeding as in the proof of Corollary 3.6, we find that (for any $\ell \ne j$), we have

$$E((u_{N,M}(j) - u_{N-1,M}(j))^2) \le \frac{K\lambda}{b^4}(\frac{1}{1-b}ER^2_{N-1,M}(\ell,j) + \frac{1}{N}),$$

and thus (3.29) follows from the fact that

$$ER^2_{N,M}(\ell,j) \le \frac{K(\alpha)}{N},$$

as shown by (3.4), (1.12). ∎

4. Empirical measures.

The purpose of this section is to show that the empirical measures

(4.1)
$$\mu_{N,M,u} = \frac{1}{M}\sum_{j\le M}\delta_{u_{N,M}(j)}$$

and

(4.2)
$$\mu_{N,M,z} = \frac{1}{N}\sum_{i\le N}\delta_{z_{N,M}(i)}$$

are essentially non random. To do this, we consider an independent copy $u'_{N,M}(j)$ of the variable $u_{N,M}(j)$ (that is, the sequence corresponding to $u_{N,M}(j)$ when the r.v. $a_{i,j}$ are replaced by an independent family $a'_{i,j}$), and we set

$$\mu'_{N,M,u} = \frac{1}{M}\sum_{j\le M}\delta_{u'_{N,M}(j)}.$$

We define $\mu'_{N,M,z}$ similarly. We will prove the following.

Proposition 4.1. *Under the conditions of Theorem 1.1, we have*

$$\lim_{N\to\infty} E\Delta(\mu_{N,M,u}, \mu'_{N,M,u}) = 0 \tag{4.3}$$

$$\lim_{N\to\infty} E\Delta(\mu_{N,M,z}, \mu'_{N,M,z}) = 0. \tag{4.4}$$

There and below, Δ denotes the square of Wasserstein's distance. Given two probability measures μ, ν on $\mathbb{R}$ it is defined as

$$\Delta(\mu, \nu) = \inf E(X - Y)^2,$$

where the infimum is taken over all the couples (X, Y) such that X has law μ and Y has law ν. Of course in (4.3), (4.4) we could use other distances ; but the use of Δ will be very convenient. We will use the duality formula

$$\Delta(\mu, \nu) = \sup(\int f d\mu - \int g d\nu) \tag{4.5}$$

where the supremum is taken over all the couples (f, g) of measurable functions such that

$$\forall x, y \in \mathbb{R}, f(x) - g(y) \leq (x - y)^2.$$

We will use (4.5) only when μ and ν have compact support, in which case the proof of (4.5) takes only a few lines (see [T1] p. 924.) The formula (4.5) lets us replace Δ by a supremum of quantities like $\int f d\mu - \int g d\nu$, which are much easier than Δ to evaluate by induction over N, M. In order to evaluate efficiently the expectation of a supremum of r.v., we will however need to consider higher moments of these r.v, and this will create complications. Another source of complications is that, while we know that $0 \leq u_{N,M}(j) \leq 1$, the numbers $z_{N,M}(i)$ can conceivably be very large, and we will need to prove some boundedness property of these quantities.

It would be very nice to replace (4.5) and (4.6) by a statement with a clean rate of convergence, such as (1.12). We could not do this. A desirable result in this direction is stated in Conjecture 5.8

Our first task will be to prove a boundedness property for the quantities $z_{N,M}$. We will denote by $K(\alpha, \lambda)$ a number depending only upon α, λ, but not upon N. This quantity need not be the same at each occurrence.

Lemma 4.2. *We have (under the conditions of Theorem 1.1)*

$$\forall a > 0, \forall b > 0, \lim_{N\to\infty} P\Big(\frac{1}{N}\sum_{k\le N} z^2_{N,M}(k)1_{\{a\le z_{N,M}(k)\le b\}} \ge \frac{K(\alpha,\lambda)}{a}\Big) = 0. \tag{4.6}$$

Proof. We will prove that

$$\forall b > 0, \forall t > 0, \limsup_{N\to\infty} E\exp\frac{t}{N}\sum_{k\le N}(z_{N,M}(k)\wedge b)^3 \le \exp t\ K(\alpha,\lambda). \tag{4.7}$$

This implies that

$$\lim_{N\to\infty} P\Big(\frac{1}{N}\sum_{k\le N}(z_{N,M}(k)\wedge b)^3 \ge K(\alpha,\lambda)\Big) = 0,$$

and (4.6). To prove (4.7), we consider

$$\theta_{N,M}(t) = E\exp\frac{t}{N}\sum_{k\le N}(z_{N,M}(k)\wedge b)^3,$$

so that, by symmetry,

$$\theta'_{N,M}(t) = E\left(z_{N,M}(N)\wedge b)^3\exp\frac{t}{N}\sum_{k\le N}(z_{N,M}(k)\wedge b)^3\right).$$

We appeal to (3.31) to get

$$\limsup_{N\to\infty}\theta'_{N,M}(t) \le \limsup_{N\to\infty} E\left(z_{N,M}(N)^3\exp\frac{t}{N}\sum_{k\le N-1}(z_{N-1,M}(k)\wedge b)^3\right). \tag{4.8}$$

(It is here that the truncation at level b is useful). We now appeal to (2.10). We observe that $z_{N-1,M}$ is independent of $c_{N,\ell}$. We integrate first in these, using Lemma 3.4 and the argument of (3.15) to see that the right-hand side of (4.8) is at most

$$K(\alpha,\lambda)\limsup_{N\to\infty} E\left(\exp\frac{t}{N}\sum_{k\le N-1}(z_{N-1,M}(k)\wedge b)^3\right) \le K(\alpha,\lambda)\limsup_{N\to\infty}\theta_{N,M}(t), \tag{4.9}$$

using (3.31) again.

Thus, we have (since $\theta_{N,M}(0) = 1$)

$$\limsup_{N\to\infty}\theta_{N,M}(t) \le 1 + \int_0^t \limsup_{N\to\infty}\theta'_{N,M}(x)dx \le 1 + K(\alpha,\lambda)\int_0^t \limsup_{N\to\infty}\theta_{N,M}(x)dx := \xi(t)$$

where we use (4.8), (4.9) in the last inequality. This means that

$$\xi'(t) \le K(\alpha,\lambda)\xi(t),$$

so that, since $\xi(0)=1$,

$$\xi(t) \le \exp tK(\alpha,\lambda). \ \square$$

Lemma 4.3. *We have*

$$E\Big(\frac{1}{N}\sum_{k\le N} z_{N,M}^2(k)1_{\{z_{N,M}(k)\ge b\}}\Big) \le \frac{K(\alpha,\lambda)}{b}.$$

Proof. It suffices to observe (as should be now obvious) that

$$E(z_{N,M}^3(N)) \le K(\alpha,\lambda). \ \square$$

There is a significant difference between the previous two results. Lemma 4.2 provides an excellent control of the "intermediate" values of $z_{N,M}$, while Lemma 4.3 provides a weak control of the large values.

Lemma 4.4. *We have*

$$\Delta\Big(\frac{1}{N}\sum_{i\le N}\delta_{x_i}, \frac{1}{N}\sum_{i\le N}\delta_{y_i}\Big) = \inf_{\sigma}\frac{1}{N}\sum_{i\le N}(x_i-y_{\sigma(i)})^2 \tag{4.10}$$

where the infimum is taken over all permutations σ *of* $\{1,...,N\}$.

Proof. The inequality $\le$ is obvious. For the converse inequality it should be obvious that the left-hand side of (4.10) is

$$\frac{1}{N}\inf\sum_{i,j\le N} a_{ij}(x_i-y_j)^2,$$

where the infimum is taken over all bistochastic matrices (a_{ij}). The infimum is obtained at an extreme point, and this extreme point is a permutation matrix. $\square$

Considering two functions f,g on $\mathbb{R}$ we set

$$U_{N,M,f,g}(t) = \exp\frac{t}{M}\Big(\sum_{j\le M} f(u_{N,M}(j)) - g(u'_{N,M}(j))\Big). \tag{4.11}$$

We set

$$\Delta_{N,M,z} = \min(2, \Delta(\mu_{N,M,z}, \mu'_{N,M,z})). \tag{4.12}$$

The use of the truncation at level 2 is to provide boundedness. We consider the function

$$\varphi_{N,M,p}(t) = E(\Delta_{N,M,z}^p U_{N,M,f,g}(t)),$$

where the dependence of the left-hand side upon f,g is implicit.

Lemma 4.5. *Assume that* $|f|,|g|\leq 1$ *and that*

$$\forall x,y,f(x)-g(y)\leq(x-y)^2. \tag{4.13}$$

Then under the conditions of Theorem 1.1, for each p*, we have*

$$\varphi'_{N,M,p}(t)\leq\frac{2}{\lambda^2}\varphi_{N,M,p+1}(t)+r_N, \tag{4.14}$$

where $\lim_{N\to\infty}r_N=0$.

Proof. Using the symmetry between sites, we have

$$\varphi'_{N,M,p}(t)=E\left(\left(f(u_{N,M}(M))-g(u'_{N,M}(M))\right)\Delta^p_{N,M,z}U_{N,M,f,g}(t)\right).$$

Using Proposition 3.7, we then see that

$$\varphi'_{N,M,p}(t)\leq E\left(\left(f(u_{N,M}(M))-g(u'_{N,M}(M))\right)\Delta^p_{N,M-1,z}U_{N,M-1,f,g}(t)\right)+r_N, \tag{4.15}$$

where $r_n\to 0$. (This step uses the fact that $\Delta_{N,M-1,z},f,g$ are bounded). We now appeal to (2.9). Since $z_{N,M-1},u_{N,M-1}$ are independent of $c_{k,M}$, the left-hand side of (4.15) is at most

$$E(A_{N,M}\Delta^p_{N,M-1,z}U_{N,M-1,f,g}(t))+r_N,$$

where

$$A_{N,M}=E_c(f(\frac{1}{1+\sum_{k\leq N}z_{N,M-1}(k)c_{k,M}})-g(\frac{1}{1+\sum_{k\leq N}z'_{N,M-1}(k)c_{k,M}})) \tag{4.16}$$

and where E_c denotes expectation in $c_{k,M},c'_{k,M}(k\leq N)$ only. Given a permutation σ of $\{1,...,N\}$ we have

$$A_{N,M}=E_c(f(\frac{1}{1+\sum_{k\leq N}z_{N,M-1}(k)c_{k,M}})-g(\frac{1}{1+\sum_{k\leq M}z'_{N,M-1}(\sigma(k))c_{k,M}})).$$

Using (4.13), we have

$$A_{N,M}\leq E_c((\sum_{k\leq N}c_{k,M}(z_{N,M-1}(k)-z'_{N,M-1}(\sigma(k)))^2).$$

Using Lemma 3.2, we have (for $\lambda\leq 1$)

$$A_{N,M}\leq\frac{2}{\lambda^2}\frac{1}{N}\sum_{k\leq N}(z_{N,M-1}(k)-z'_{N,M-1}(\sigma(k)))^2.$$

In this inequality, σ is arbitrary, so that Lemma 4.4 shows that

$$A_{N,M} \leq \frac{2}{\lambda^2}\Delta(\mu_{N,M-1,z}, \mu'_{N,M-1,z}).$$

On the other hand, since $|f|, |g| \leq 1$, it is obvious that $A_{N,M} \leq 2$, so that,

$$A_{N,M} \leq \frac{2}{\lambda^2}\Delta_{N,M-1,z},$$

and thus the left-hand side of (4.15) is at most

$$\frac{2}{\lambda^2}E(\Delta^{p+1}_{N,M-1,z}U_{N,M-1,f,g}(t)) + r_N,$$

and appealing again to Proposition 3.7 yields the result. □

Proposition 4.6. *Under the conditions of Lemma 4.5, we have*

$$\limsup_{N\to\infty} E(U_{N,M,f,g}(t)) \leq \limsup_{N\to\infty} E(\exp\frac{2t}{\lambda^2}\Delta_{N,M,z}). \tag{4.17}$$

Proof. We prove by induction over $p \geq 0$ that

$$\begin{aligned}\varphi_{N,M,0}(t) \leq &\sum_{q=0}^{p}(\frac{2}{\lambda^2}t)^q\frac{1}{q!}E(\Delta^q_{N,M,z})\\ &+ (\frac{2}{\lambda^2})^{p+1}\int_0^t \frac{1}{p!}(t-x)^p\varphi_{N,M,p+1}(x)dx + r_{N,p}(t),\end{aligned} \tag{4.18}$$

where $\lim_{N\to\infty} r_{N,p}(t) = 0$. For $p = 0$, this follows from the fact that

$$\varphi_{N,M,0}(t) = 1 + \int_0^t \varphi'_{N,M,0}(x)dx$$

and (4.14); while the induction step follows by integration by parts and (4.14). It follows from (4.18) that

$$\limsup_{N\to\infty}\varphi_{N,M,0}(t) \leq \limsup_{N\to\infty} E\exp(\frac{2}{\lambda^2}t\Delta_{N,M,z})+(\frac{2}{\lambda^2})^{p+1}\int_0^t \frac{1}{p!}(t-x)^p \limsup_{N\to\infty}\varphi_{N,M,p+1}(x)dx,$$

from which (4.17) follows as $p \to \infty$, since $\varphi_{N,M,p+1}(x) \leq 4^p e^{4x}$. □

Considering again two functions f, g on $\mathbb{R}$, we set

$$Z_{N,M,f,g}(t) = \exp\frac{t}{N}\sum_{i\leq N}(f(z_{N,M}(i)) - g(z_{N,M}(i))) \tag{4.19}$$

and we set

$$\Delta_{N,M,u} = \Delta(\mu_{N,M,u}, \mu'_{N,M,u}). \tag{4.20}$$

We observe that $\Delta_{N,M,u} \le 1$, because $0 \le u_{N,M}(j) \le 1$.

We consider the function

$$\psi_{N,M,p}(t) = E(\Delta^p_{N,M,u} Z_{N,M,f,g}(t)).$$

Lemma 4.7. *We assume f, g bounded, and we assume (4.13). Then, under the condition of Theorem 1.1, for each t, we have*

$$\psi'_{N,M,p}(t) \le \frac{K\lambda^3}{\alpha^8}\psi_{N,M,p+1}(t) + r_N, \tag{4.21}$$

where $\lim_{N\to\infty} r_N = 0$.

Proof. It is essentially identical to the proof of Lemma 4.5. Rather than (4.16), we must now deal with

$$B_{N,M} = E_c\Big(f\Big(\frac{1}{\sum_{\ell\le M} u_{N-1,M}(\ell)c_{N,\ell}}\Big) - g\Big(\frac{1}{\sum_{\ell\le M} u'_{N-1,M}(\sigma(\ell))c'_{N,\ell}}\Big)\Big). \tag{4.22}$$

Using (4.13), we get

$$B_{N,M} \le E_c\Big(\Big(\frac{\sum_{\ell\le M} c_{N,\ell}(u_{N-1,M}(\ell) - u'_{N-1,M}(\sigma(\ell)))}{(\sum_{\ell\le M} c_{N,\ell}u_{N-1,M}(\ell))(\sum_{\ell\le M} c_{N,\ell}u'_{N-1,M}(\sigma(\ell)))}\Big)^2\Big),$$

and to control this we use (2.15) and Proposition 3.5. (Observe that $\sum c_\ell w_\ell = \sum \dot{c} w_\ell$ if $\sum w_\ell = 0$.) ∎

Proposition 4.8. *Under the conditions of Lemma 4.7, we have*

$$\limsup_{N\to\infty} E(Z_{N,M,f,g}(t)) \le \limsup_{N\to\infty} E\exp\Big(\frac{K\lambda^3}{\alpha^8} t\Delta_{N,M,u}\Big). \tag{4.23}$$

Proof. Identical to that of Proposition 4.6. ∎

We set

$$\Phi_u(t) = \limsup_{N\to\infty} E\exp t\Delta_{N,M,u} \tag{4.24}$$

$$\Phi_z(t) = \limsup_{N\to\infty} E\exp t\Delta_{N,M,z}. \tag{4.25}$$

Proposition 4.9. *Given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that*

$$\forall t \geq 0, \Phi_u(t) \leq N(\varepsilon)\Phi_z(\frac{2}{\lambda^2}t)e^{t\varepsilon}. \tag{4.26}$$

Proof. By (4.3) there exists a finite family $\mathcal{F}$ of couples (f,g) of functions satisfying (4.11) such that, given two probabilities μ, ν supported by $[0,1]$, we have

$$\Delta(\mu,\nu) \leq \varepsilon + \sup_{(f,g)\in\mathcal{F}} (\int f d\mu - \int g d\nu).$$

Since $\mu_{N,M,u}, \mu'_{N,M,u}$ are supported by $[0,1]$, we then have

$$E \exp t\Delta_{N,M,u} \leq e^{t\varepsilon} E \sup_{(f,g)\in\mathcal{F}} \exp tU_{N,M,f,g}.$$

Using the bound $E(\sup Y_\ell) \leq \sum_\ell EY_\ell$ for $Y_\ell \geq 0$, (4.17) yields the result, with $N(\varepsilon) = \mathrm{card}\mathcal{F}$. ∎

Proposition 4.10. *Given $\varepsilon > 0$, there exists $M(\varepsilon)$ such that*

$$\forall t \geq 0, \Phi_z(t) \leq M(\varepsilon)\Phi_u(\frac{K\lambda^3}{\alpha^8}t)e^{15t\varepsilon}. \tag{4.27}$$

Proof. The extra difficulty here compared to Proposition 4.9 is that $z_{N,M}(k)$ is not bounded.

According to (4.4), we can find $a > 0$ such that

$$\forall b > 0, \lim_{N\to\infty} P(\frac{1}{N}\sum_{k\leq N} z^2_{N,M}(k)1_{\{a\leq z_{N,M}(k)\leq b\}} \geq \varepsilon) = 0. \tag{4.28}$$

We then fix a finite family $\mathcal{F}$ of bounded functions satisfying (4.11), such that, given any two probability measures μ,ν with support in $[0,a]$, we have

$$\Delta(\mu,\nu) \leq \varepsilon + \sup_{(f,g)\in\mathcal{F}} (\int f d\mu - \int g d\nu). \tag{4.29}$$

Only the values of f, g on $[0,a]$ matter there. Condition (4.11) still hold if we replace the couple (f,g) by the couple (f^a, g^a), where

$$f^a(x) = f(\min(x,a)),$$

and where g^a is defined similarly. That is, we can assume

$$f(x) = f(\min(x,a)), g(x) = g(\min(x,a)), \tag{4.30}$$

whenever $(f,g)\in\mathcal{F}$. For a probability measure μ on $\mathbb{R}^+$, let us denote μ^a its image under the map $x\to\min(x,a)$. Then by (4.29), for any two probabilities μ,ν on $\mathbb{R}^+$,

$$\begin{aligned}\Delta(\mu^a,\nu^a) &\le \varepsilon+\sup_{(f,g)\in\mathcal{F}}\Big(\int f d\mu^a-\int g d\nu^a\Big)\\ &=\varepsilon+\sup_{(f,g)\in\mathcal{F}}\Big(\int f d\mu-\int g d\nu\Big),\end{aligned}$$

using (4.30). Now, since Δ is the square of a distance,

$$\Delta(\mu,\nu)\le 3(\Delta(\mu,\mu^a)+\Delta(\mu^a,\nu^a)+\Delta(\nu,\nu^a)),$$

and, obviously

$$\Delta(\mu,\mu^a)\le\int(x-\min(x,a))^2 d\mu(x).$$

Thus,

$$\begin{aligned}\Delta(\mu_{N,M,z},\mu'_{N,M,z}) &\le 3\varepsilon+3\sup_{(f,g)\in\mathcal{F}}\frac{1}{N}\sum_{i\le N}(f(z_{N,M}(i))-g(z'_{N,M}(i)))\\ &+\frac{3}{N}\sum_{i\le N}z^2_{N,M}(i)1_{\{z_{N,M}(i)\ge a\}}+\frac{3}{N}\sum_{i\le N}z'^2_{N,M}(i)1_{\{z'_{N,M}(i)\ge a\}}.\end{aligned}$$

Thus, for any $b>0$, we have

$$\Delta_{N,M,z}\le\min(2,3\varepsilon+3S+3U+3U'+3V+3V'),$$

where

$$\begin{aligned}S&=\sup_{(f,g)\in\mathcal{F}}\frac{1}{N}\sum_{i\le N}(f(z_{N,M}(i))-g(z'_{N,M}(i))),\\ U&=\frac{1}{N}\sum_{i\le N}z^2_{N,M}(i)1_{\{a<z_{N,M}(i)\le b\}}\\ V&=\frac{1}{N}\sum_{i\le N}z^2_{N,M}(i)1_{\{z_{N,M}(i)\ge b\}},\end{aligned}$$

and U',V' are defined similarly, replacing z by z'. Thus, since $\Delta_{N,M,z}\le 2$, we have

$$\begin{aligned}E\exp t\Delta_{N,M,z} &\le \exp 15t\varepsilon E\left(\sup_{(f,g)\in\mathcal{F}}(Z_{N,M,f,g}(3t))^3\right)\\ &+e^{2t}(P(U\ge\varepsilon)+P(U'\ge\varepsilon)+P(V\ge\varepsilon)+P(V'\ge\varepsilon)).\end{aligned}\tag{4.31}$$

From (4.7), we have

$$P(V\ge\varepsilon)=P(V'\ge\varepsilon)\le\frac{K(\alpha,\lambda)}{\varepsilon b},$$

so that, if $M(\varepsilon) = \text{card}\mathcal{F}$, we see from (4.31) and Proposition 4.8 that

$$\Phi_z(t) \leq M(\varepsilon)e^{15t\varepsilon}\Phi_u(\frac{K\lambda^3}{\alpha^8}t) + 2e^{2t}\frac{K(\alpha,\lambda)}{\varepsilon b},$$

and the result follows by taking $b \to \infty$. ∎

Proof of Proposition 4.1. If we combine (4.26) and (4.27), we get

$$\forall t > 0, \Phi_u(t) \leq K(\varepsilon)e^{A\varepsilon t}\Phi_u(\frac{K\lambda}{\alpha^8}t),$$

where $A = 1 + 30/\lambda^2$. If $\lambda \geq \alpha^8/K$, we then get

$$\forall t > 0, \Phi_u(t) \leq K(\varepsilon)e^{A\varepsilon t}\Phi_u(\frac{t}{2}).$$

Since $\Phi_u(1) \leq e$, we get by iteration that for all $k \geq 1$,

(4.32)
$$\Phi_u(2^k) \leq (K(\varepsilon))^k e^{A\varepsilon 2^k}.$$

Now, by Chebishev inequality,

$$P(\Delta_{N,M,u} \geq x) \leq e^{-2^k x} E \exp 2^k \Delta_{N,M,u},$$

so that, using (4.32)

$$\limsup_{N\to\infty} P(\Delta_{N,M,u} \geq x) \leq (K(\varepsilon))^k e^{(A\varepsilon - x)2^k}.$$

Taking $x = (A+1)\varepsilon$, and letting $k \to \infty$, we get that

$$\limsup_{N\to\infty} P(\Delta_{N,M,u} \geq (A+1)\varepsilon) = 0.$$

Since $\Delta_{N,M,u} \leq 1$, and since ε is arbitrary, this proves (4.3). The proof of (4.4) is similar. ∎

Let us denote by $\mathcal{L}(X)$ the law of a r.v X.

Theorem 4.11. *Under the conditions of Theorem 1.1, we have*

$$\lim_{N\to\infty} E\Delta(\mu_{N,M,u}, \mathcal{L}(u_{N,M})) = 0$$

$$\lim_{N\to\infty} E\Delta(\mu_{N,M,z}, \mathcal{L}(z_{N,M})) = 0.$$

Proof. This is a consequence of the general fact that if ν is a random probability,

(4.33)
$$E\Delta(\mu, \nu) \geq \Delta(\mu, E\nu),$$

and that $E\mu_{N,M,u'} = \mathcal{L}(u_{N,M})$. Thus Theorem 4.11 follows from Proposition 4.1 by "integrating in u' inside Δ rather then outside". □

5. Proof of Theorem 1.1 ; some consequences.

The distribution of $\sum_\ell u(\ell) c_{N,\ell}$ (where $(u(\ell))_{\ell \le M}$ are given numbers) depends only upon the probability $M^{-1} \sum_{\ell \le M} \delta_{u(\ell)}$. Thus we see from (2.10) and Theorem 4.11 that $\mathcal{L}(z_{N,M})$ is essentially determined by $\mathcal{L}(u_{N-1,M})$. In a similar fashion, $\mathcal{L}(u_{N,M})$ is essentially determined by $\mathcal{L}(z_{N,M-1})$. Since $\mathcal{L}(u_{N,M}) \approx \mathcal{L}(u_{N-1,M-1})$ by Proposition 3.6, we see that $\mathcal{L}(u_{N,M})$ must be a (nearly) fixed point of a certain transformation.

We start with the following technical fact (the proof of which is probably better skipped at first reading, since the real action starts only Proposition 5.3).

Lemma 5.1. *Consider independent r.v. a, a_ℓ, uniform over $[0,1]$. Consider an integer $R \ge 1$, consider $\nu > 0$ and*

$$c = \exp(-\nu a); c' = \sum_{\ell \le R} \exp(-\nu R a_\ell). \tag{5.1}$$

Then we can find a joint realization (X, X') of c, c' such that

$$E(X - X')^2 \le \frac{K}{\nu^2}; E(X - X')^4 \le \frac{K}{\nu^2}. \tag{5.2}$$

Comment. This will be used for $\nu = \lambda N$.

Proof. For $0 \le t \le 1$, we have

$$P(c \ge t) = \min(1, \frac{1}{\nu} \log \frac{1}{t})$$

$$P(\exp(-\nu R a_\ell) \ge t) = \min(1, \frac{1}{\nu R} \log \frac{1}{t}).$$

Since $c' \ge t$ provided one of the summands is at least t, we have, by independence,

$$P(c' \ge t) \ge 1 - (1 - \min(1, \frac{1}{\nu R} \log \frac{1}{t}))^R := \varphi(t).$$

Since $(1-x)^R \ge 1 - Rx$, we have

$$\varphi(t) \le \min(R, \frac{1}{\nu} \log \frac{1}{t})$$

and, since $\varphi(t) \leq 1$ we have in fact

$$\varphi(t) \leq \min(1, \frac{1}{\nu} \log \frac{1}{t}) = P(c \geq t). \tag{5.3}$$

Since

$$x \leq 1 \Rightarrow (1 - \frac{x}{R})^R \leq 1 - x + x^2,$$

we have

$$\frac{1}{\nu} \log \frac{1}{t} \leq 1 \Rightarrow \varphi(t) \geq \frac{1}{\nu} \log \frac{1}{t} - (\frac{1}{\nu} \log \frac{1}{t})^2,$$

and since $\varphi(t) \geq 0$, we have

$$0 \leq P(c \geq t) - \varphi(t) \leq (\frac{1}{\nu} \log \frac{1}{t})^2, \tag{5.4}$$

even when $\nu^{-1} \log(1/t) \geq 1$. We have

$$\int_0^1 |P(c \geq t) - P(c' \geq t)|dt \leq \int_0^1 |P(c \geq t) - \varphi(t)|dt + \int_0^1 |P(c' \geq t) - \varphi(t)|dt \tag{5.5}$$

$$= \int_0^1 |P(c \geq t) - \varphi(t)|dt + \int_0^1 (P(c' \geq t) - \varphi(t))dt$$

$$\leq 2 \int_0^1 |P(c \geq t) - \varphi(t)|dt + \int_0^1 (P(c' \geq t) - P(c \geq t))dt.$$

Since

$$\int_0^1 P(c \geq t)dt = Ec, \int_0^1 P(c' \geq t)dt = Ec',$$

it follows readily that the left-hand side of (5.5) is at most K/ν^2. There exists two r.v. X, X', with $\mathcal{L}(X) = \mathcal{L}(c), \mathcal{L}(X') = \mathcal{L}(c')$ and

$$E|X - X'| = \int_0^1 |P(c \geq t) - P(c' \geq t)|dt.$$

(For example, if we define ψ by

$$P(c \geq t) = P(c' \geq \psi(t)),$$

the couple $(\psi(t), t)$ on the probability space $(\mathbb{R}, \mathcal{L}(c'))$ works.)

In particular, since $0 \leq X \leq 1$,

$$E|X - \min(2, X')|^2 \leq \frac{K}{\nu^2}; E|X - \min(2, X')|^4 \leq \frac{K}{\nu^2},$$

i.e. $\Delta(\mathcal{L}(c), \mathcal{L}(\min(2, c')) \leq K/\nu^2$. It remains to show that $E((c' - \min(2, c'))^4) \leq K/\nu^4$, or even $E((c' 1_{\{c' \geq 2\}})^4) \leq K/\nu^4$. This follows from simple tail estimates for c'. $\square$

Lemma 5.2. *If a is uniform over* $[0,1]$ *we have, for* $p \geq 1$,

$$E|e^{-\nu a} - e^{-\nu' a}|^p \leq K|\frac{1}{\nu} - \frac{1}{\nu'}|.$$

Proof. Since $|e^{-\nu a} - e^{-\nu' a}| \leq 1$, we have

$$\begin{aligned} E|e^{-\nu a} - e^{-\nu' a}|^p &\leq E|e^{-\nu a} - e^{-\nu' a}| \\ &= |Ee^{-\nu a} - Ee^{-\nu' a}| \end{aligned}$$

because $e^{-\nu a} \geq e^{-\nu' a}$ if $\nu \leq \nu'$. □

Proposition 5.3. *Given a number* $\alpha > 0$, *there is a number* $\lambda(\alpha) > 0$ *with the following property. If* $\lambda \leq \lambda(\alpha)$, *to each probability measure* μ *on* $[0,1]$, *such that* $\int x d\mu(x) \geq \alpha/2$, *we can associate a probability measure* $A(\mu)$ *on* $\mathbb{R}^+$ *such that the following occurs.*

(5.6)
$$\text{If } \frac{1}{M}\sum_{\ell \leq M} u(\ell) \geq \frac{\alpha}{2}, \text{then}$$

$$\Delta(A(\mu), \mathcal{L}(\frac{1}{\sum_{\ell \leq M} c_{N,\ell} u(\ell)})) \leq K(\alpha,\lambda)[\frac{1}{N} + |\frac{M}{N} - 1 - \alpha|]$$
$$+\frac{K\lambda^2}{\alpha^8}|\int x d\mu(x) - \frac{1}{M}\sum_{\ell \leq M} u(\ell)| + \frac{K\lambda^3}{\alpha^8}\Delta(\mu, \frac{1}{M}\sum_{\ell \leq M}\delta_{u(\ell)})$$

Moreover,

(5.7)
$$\Delta(A(\mu), A(\mu')) \leq \frac{K\lambda^2}{\alpha^8}|\int x d\mu(x) - \int x d\mu'(x)| + \frac{K\lambda^3}{\alpha^8}\Delta(\mu, \mu').$$

The proposition asserts only the existence of the operator A. One can show easily that this operator is unique. In fact, one can show that $A(\mu)$ is the law of $(\sum_{i \geq 1}\exp(-\lambda\xi_i)X_i)^{-1}$, where the r.v. X_i are i.i.d with distribution μ, and where the variables ξ_i are the arrival times of a Poisson point process of intensity measure 1, independent of the variables X_i. This interpretation however does not seem to make the proof any easier.

Proof. Consider M', N', numbers $(u'(\ell))_{\ell \leq M'}$ with $M'^{-1}\sum_{\ell \leq M'} u'(\ell) \geq \alpha/2$. We will prove that

(5.8)
$$\Delta(\mathcal{L}(\frac{1}{\sum_{\ell \leq M} c_{N,\ell} u(\ell)}), \mathcal{L}(\frac{1}{\sum_{\ell \leq M'} c_{N',\ell} u'(\ell)}))$$
$$\leq K(\alpha,\lambda)(\frac{1}{N} + \frac{1}{N'} + |\frac{M}{N} - 1 - \alpha| + |\frac{M'}{N'} - 1 - \alpha|)$$

$$+\frac{K\lambda^2}{\alpha^8}\Big|\frac{1}{M}\sum_{\ell\le M}u(\ell)-\frac{1}{M'}\sum_{\ell\le M'}u'(\ell)\Big|+\frac{K\lambda^3}{a^8}\Delta\Big(\frac{1}{M}\sum_{\ell\le M}\delta_{u(\ell)},\frac{1}{M'}\sum_{\ell\le M'}\delta_{u'(\ell)}\Big).$$

A cluster point argument will prove (5.6) and the existence of $A(\mu)$, from which (5.7) follows. The main difficulty in proving (5.8) is that we can have $M\neq M'$, and the purpose of Lemma 5.1 is to address this, by showing that one can replace both M and M' by MM'. Indeed using this lemma for $\nu=\lambda N, R=M'$, we see that we can find independent variables $(c'_\ell)_{\ell\le M}$, each the sum of M' independent copies of $\exp(-\lambda NM'a)$ (where a is uniform over $[0,1]$) and such that for each $\ell\le M$,

$$E(c_{N,\ell}-c'_\ell)^2\le\frac{K}{\lambda^2N^2};E(c_{N,\ell}-c'_\ell)^4\le\frac{K}{\lambda^2N^2}. \tag{5.9}$$

We then write

$$E\Big(\Big(\frac{1}{\sum_{\ell\le M}c_{N,\ell}u(\ell)}-\frac{1}{\sum_{\ell\le M'}c'_\ell u(\ell)}\Big)^2\Big) \tag{5.10}$$

$$\le\Big(E\big(\sum_{\ell\le N}(c_{N,\ell}-c'_\ell)u(\ell)\big)^4\Big)^{1/2}\Big(E\Big(\frac{1}{(\sum_{\ell\le M}c_{N,\ell}u(\ell)}\Big)^8\Big)^{1/4}\Big(E\Big(\frac{1}{\sum_{\ell\le M}c'_\ell u(\ell)}\Big)^8\Big)^{1/4},$$

and we use Lemma 3.4 (as in the proof of Proposition 3.5) to obtain a bound $K(\lambda,\alpha)/N$ for (5.10). This allows to replace M by MM'. Similarly, we replace M' by MM'.

We have reduced the problem to bound the left-hand side of (5.8) when M' and M are replaced by MM' i.e. we are required to bound

$$\Delta\Big(\mathcal{L}\Big(\frac{1}{\sum_{\ell\le MM'}c_\ell u(\ell)}\Big),\mathcal{L}\Big(\frac{1}{\sum_{\ell\le MM'}c'_\ell u'(\ell)}\Big)\Big).$$

There $c_\ell\overset{\mathcal{D}}{=}\exp(-\lambda NM'a), c'_\ell\overset{\mathcal{D}}{=}\exp(-\lambda N'Ma)$. Using Lemma 5.2, and proceeding as before, we see that we can replace c'_ℓ by c_ℓ, making an error at most $K(\alpha,\lambda)|M/N-M'/N'|$. If we knew that $\sum_\ell u(\ell)=\sum_\ell u'(\ell)$, we would be finished by Proposition 3.5 and Lemma 4.4. But we simply reduce to (3.11) by writing

$$\Big(\sum_\ell\dot{c}_\ell w_\ell\Big)^2\le2\Big(\sum_\ell\dot{c}_\ell w_\ell\Big)^2+2(Ec_\ell)^2\Big(\sum_\ell w_\ell\Big)^2.\ \square$$

Proposition 5.4. *We have*

$$\lim_{N\to\infty}\Delta(\mathcal{L}(z_{N,M}),A(\mathcal{L}(u_{N,M})))=0. \tag{5.11}$$

Proof. Using (4.33), (2.10) it is enough to prove that

$$\lim_{N\to\infty}E\Delta\Big(\mathcal{L}_c\Big(\frac{1}{\sum_{\ell\le M}u_{N-1,M}(\ell)c_{N,\ell}}\Big),A(\mathcal{L}(u_{N,M}))\Big)=0,$$

where $\mathcal{L}_c$ denotes the law at $(u_{N-1,M}(\ell))$ given. We can replace $\mathcal{L}(u_{N,M})$ by $\mathcal{L}(u_{N-1,M})$ thanks to (5.7) and Proposition 3.7. The conclusion follows from (5.6) and Theorem 4.11.

Proposition 5.5. *Given $\lambda > 0$, to each probability measure μ on $\mathbb{R}^+$ we can associate a probability measure $B(\mu)$ on $[0,1]$ such that the following occurs, for any numbers $z(k) \geq 0, k \leq N$:*

$$\Delta(B(\mu), \mathcal{L}(\frac{1}{1+\sum_{k\leq N} c_{k,M} z(k)})) \leq \frac{K(\lambda)}{N} + \frac{K}{\lambda^2}\Delta(\mu, \frac{1}{N}\sum_{k\leq N}\delta_{z(k)}). \tag{5.12}$$

Moreover, for two probability measures μ, μ' on $\mathbb{R}^+$, we have

$$\Delta(B(\mu), B(\mu')) \leq \frac{K}{\lambda^2}\Delta(\mu, \mu'). \tag{5.13}$$

Proof. Similar to Proposition 5.3, (but easier) using now Lemma 3.2.

Proposition 5.6. *We have*

$$\lim_{N\to\infty} \Delta(\mathcal{L}(u_{N,M}), B(\mathcal{L}(z_{N,M}))) = 0. \tag{5.14}$$

Proof. As in Proposition 5.4. □

Theorem 5.7. *The limits*

$$\mu_u = \lim_{N\to\infty} \mathcal{L}(u_{N,M})$$

$$\mu_z = \lim_{N\to\infty} \mathcal{L}(z_{N,M})$$

exist.

Proof. Combining (5.11) and (5.14), we see that

$$\lim_{N\to\infty} \Delta(\mathcal{L}(u_{N,M}), B{\circ}A(\mathcal{L}(u_{N,M}))) = 0,$$

so that any cluster point μ of the sequence $\mathcal{L}(u_{N,M})$ is a fixed point of $B{\circ}A$, and satisfies $\int x d\mu(x) = \alpha$. But (5.7), (5.13) show that this cluster point is unique if $\lambda \leq \alpha^8/K$. Thus $\lim \mathcal{L}(u_{N,M}) := \mu_u$ exists, and, of course, $\mu_z = A(\mu_u)$.

Comment. The mysterious part of the proof is that for <u>each</u> α not too small, $B{\circ}A$ admits a fixed point μ such that $\int x d\mu(x) = \alpha$. (This is also probably true for small α.)

Conjecture 5.8. *Given any integer n, there exists a constant $K(\alpha, n)$ such that for any N, there exists independent r.v $Y_1, \cdots, Y_n$ of law μ_u with*

$$\sum_{1\leq i\leq n} (u_{N,M}(i) - Y_i)^2 \leq \frac{K(\alpha,n)}{N}.$$

Of course one can make a similar conjecture for the variables $z_{N,M}$.

Proof of (1.15). Writing

$$A_{N,M} = E \log S_{N,M},$$

we have

$$A_{N,M} - A_{N,M-1} = E \log \frac{S_{N,M}}{S_{N,M-1}} = -E \log(u_{N,M}(M-1)) \tag{5.15}$$

$$A_{N,M} - A_{N-1,M} = E \log \frac{S_{N,M}}{S_{N-1,M}} = -E \log(z_{N,M}(N)), \tag{5.16}$$

so that these quantities have limits $-\int \log x d\mu_u(x)$ and $-\int \log x d\mu_z(x)$ respectively as $N, M \to \infty, M/N \to 1+\alpha$. (Here we skip a few simple details as that better left to the reader.) We then write $A_{N,M} - A_{1,1}$ as a sum of $(N-1)$ quantities $A_{R,M(R)} - A_{R-1,M(R)} (R \leq N)$ where $M(R) = \lfloor R(1+\alpha) \rfloor$ and of about $(1+\alpha)N$ quantities

$$A_{R,M(R)-\ell} - A_{R,M(R)-\ell-1}$$

where $M(R) - \ell - 1 \geq M(R-1)$. Thus, we see that not only the limit exists in (1.15), but that it is

$$-\int \log x d\mu_u(x) - (1+\alpha) \int \log x d\mu_z(x).$$

As a conclusion, let us say a few informal words about Gibbs' measure $G_{N,M}$. We recall that

(5.17).
$$G_{N,M}(\{\sigma(i) = j\}) = c_{i,j} \frac{S_{N,M}(i;j)}{S_{N,M}}$$

In a similar manner, if $i_1 \neq i_2$ and $j_1 \neq j_2$, we have

(5.18).
$$G_{N,M}(\{\sigma(i_1) = j_1, \sigma(i_2) = j_2\}) = c_{i_1,j_1} c_{i_2,j_2} \frac{S_{N,M}(i_1, i_2; j_1, j_2)}{S_{N,M}}$$

Now we have

$$S_{N,M}(i_1, i_2; j_1, j_2) S_{N,M} \simeq S_{N,M}(i_1; j_1) S_{N,M}(i_2; j_2)$$

because both sides are nearly

$$S^3_{N,M} S_{N,M}(i_1; \emptyset) S_{N,M}(i_2; \emptyset) S_{N,M}(\emptyset; j_1) S_{N,M}(\emptyset; j_2).$$

Combining with (5.17), (5.18) this proves as announced in the introduction that

$$G_{N,M}(\{\sigma(i_1) = j_1, \sigma(i_2) = j_2\}) \simeq G_{N,M}(\{\sigma(i_1) = j_1\})G_{N,M}(\{\sigma(i_2) = j_2\}).$$

This even holds true if $j_1 = j_2$ because in that case the right-hand side is very likely to be small for all values of j_1.

REFERENCES

[**A1**] D. ALDOUS, Asymptotics in the random assignment problem, *Probab. Th. Relat. Fields* 93, 1992, 507-534.

[**A2**] D. ALDOUS, The $\zeta(2)$ limit in the random assignment problem, *Random Structures Algorithms* 18, 2001, 381-418.

[**M-P1**] M. MÉZARD, G. PARISI, Replicas and optimization, *J. Physique. Lett.* 46, 1985, L771.

[**M-P2**] M. MÉZARD, G. PARISI, Mean field equations for the matching and the traveling salesman problem *Europhys. lett.* 1986, 916.

[**M-P-V**] M. MÉZARD, G. PARISI, M.A. VIRASORO, Spin Glass Theory and Beyond, *World Scientific*, Singapore, 1987.

[**T1**] M. TALAGRAND, The transportation cost from the uniform measure to the empirical measure in dimension ≥ 3, *Ann. Probab.* 22, 1994, 919-959.

[**T2**] M. TALAGRAND, Huge random structure and mean field models for spin glasses, *Proceedings of the Congress of Mathematicians*, Documenta Mat, 1999, extra volume I, 507-536.

Equipe d'Analyse
Université Paris VI
Boîte 186
4, Place Jussieu
75252. Paris Cedex 05