ON THE HOPFIELD MODEL AT THE CRITICAL TEMPERATURE

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Abstract. We study the Hopfield model at temperature 1, when the number $M(N)$ of patterns grows a bit slower than N. We reach a good understanding of the model whenever $M(N) \le N/(\log N)^{11}$. For example, we show that if $M(N) \to \infty$, for two typical configurations $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2$, $(\sum_{i \le N} \sigma_i^1 \sigma_i^2)^2$ is close to $NM(N)$.

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1. Introduction.

The behavior of the Hopfield model with extensively many patterns is now rigorously understood rather precisely in a large domain of parameters [B-G] [T]. Much less is known at the critical temperature 1. The physicists predict that at this critical temperature, there is spin glass behavior as soon as the number $M = M(N)$ of patterns grows proportionally to N. Therefore when investigating the model at temperature 1 it seems more reasonable in a first stage to assume that $\alpha = \alpha(N) =: M(N)/N \to 0$. When M remains constant, a rather precise picture is provided by a recent work of Gentz and Lowe [G-L], that motivated the present paper. We are interested here in the case where $M(N) \to \infty$. A suitable version of the result of [G-L] (Theorem 1.1 below) remains valid if $M(N)$ does not grow too fast; apparently the critical rate of growth is $N^{1/3}$. The main thrust of the present paper is that we succeed in getting rather precise information when $M(N) = N^\delta$ and $\delta < 1$. It is in fact likely that our result continues to hold under the condition $M(N) = o(N)$, but serious technical difficulties arise when $\alpha = M(N)/N$ goes to zero too slowly, and the rather small gap remaining does not seem to warrant a major effort at this stage. This would also ruin the main appeal of the present work, which is to present a rather non trivial situation with proofs that are rather simple (at least by the current standards of the area). We now describe the model and explain our results in detail.

We consider independent random variables $(\eta_{i,k})_{i\le N,k\le M}$ with $P(\eta_{i,k} = 1) = P(\eta_{i,k} = -1) = 1/2$, and to a spin configuration $\boldsymbol{\sigma} \in \Sigma_N = \{-1,1\}^N$ we associate its Hamiltonian

$$H_{N,M}(\boldsymbol{\sigma}) = -\frac{1}{2N}\sum_{k=1}^{M}\Big(\sum_{i\le N}\eta_{i,k}\sigma_i\Big)^2 = -\frac{N}{2}\sum_{k=1}^{M} m_k^2 \tag{1.1}$$

where $m_k = m_k(\boldsymbol{\sigma}) = N^{-1}\sum\limits_{i\le N}\eta_{i,k}\sigma_i$. The quantities $m_k(\boldsymbol{\sigma})$ are called the *overlaps*. We are interested in the Gibbs measure $G_{N,M}$ associated to $H_{N,M}$ at inverse temperature 1, that is, in the (random) probability measure on Σ_N given by

$$G_{N,M}(\{\boldsymbol{\sigma}\}) = Z_{N,M}^{-1}\exp -H_{N,M}(\boldsymbol{\sigma}) \tag{1.2}$$

where $Z_{N,M}$ is the normalization factor $\sum \exp -H_{N,M}(\boldsymbol{\sigma})$, for a summation over all configurations.

Since the Hamiltonian (1.1) is defined in terms of the overlaps only, it is rather natural to study the measure $G' = G'_{N,M}$ defined as the image of $G_{N,M}$ on $\mathbb{R}^M$ under the map $\boldsymbol{\sigma} \to (m_k(\boldsymbol{\sigma}))_{k\le M}$.

A very convenient tool is the Hubbard-Stratonovich transform, that is convolution by the gaussian probability γ on $\mathbb{R}^M$ of density proportional to $\exp(-N\|\boldsymbol{z}\|^2/2)$.

It has been used in much of the literature dealing with the Hopfield model. We will denote by $\overline{G}$ the convolution $\overline{G} = G' * \gamma$. We are interested in G', not $\overline{G}$, and $\overline{G}$ is only a technical tool. But $\overline{G}$ and G' are closely related, because γ is sharply concentrated on a ball of radius $\sqrt{\alpha}$. We will always be in the case where $\alpha \to 0$, and it will turn out that the natural scale at which to look at G' is $\alpha^{1/4}$, so that even at this scale $\overline{G}$ is a good approximation of G' (since $\sqrt{\alpha} \ll \alpha^{1/4}$ for α small). Let us note in particular that if f is a Lipschitz function on $\mathbb{R}^M$, of Lipschitz constant ≤ 1 then

$$|\int f(\boldsymbol{z})dG'(\boldsymbol{z}) - \int f(\boldsymbol{z})d\overline{G}(z)| \leq L\sqrt{\alpha}. \tag{1.3}$$

There and throughout the paper, L denotes a universal constant, not necessarily the same at each occurrence.

The reason why $\overline{G}$ is so useful is that (by a simple calculation) it has a density proportional to $\exp -\psi(\boldsymbol{z})$, where

$$\psi(\boldsymbol{z}) = \frac{N\|\boldsymbol{z}\|^2}{2} - \sum_{i\leq N} \log \operatorname{ch} \boldsymbol{\eta}_i \cdot \boldsymbol{z} \tag{1.4}$$

for $\boldsymbol{\eta}_i = (\eta_{i,k})_{k\leq M}, \boldsymbol{\eta}_i \cdot \boldsymbol{z} = \sum_{k\leq M} \eta_{i,k} z_k.$

Even though the expression for ψ is formally simple, the behavior of this function is not so clear.

Central to the paper is the fact that for many purposes, we can replace ψ by the more explicit function ψ_1 given by (1.5) below. Despite its technical nature, we state it as a theorem, in homage to the paper [G-L] that very directly motivated it. The precise way to use this result will become apparent only gradually, but the overall philosophy is simple: We can replace the study of $\overline{G}$ by that of a more explicit probability. It should also be pointed out that the main reason we succeed in covering a much wider range of α than in [G-L], [G-L2] is that our formulation of the idea of replacing ψ by an approximation ψ_1 better preserves the balance between two conflicting needs: the need the make ψ_1 explicit enough to be usable, and the need to have an approximation valid in a wide enough range.

Theorem 1.1. *Consider the probability $\overline{G}_1$ on $\mathbb{R}^M$ of density proportional to $\exp -\psi_1(\boldsymbol{z})$, where*

$$\begin{aligned}\psi_1(\boldsymbol{z}) = \frac{N}{2}\sum_{k<\ell} z_k^2 z_\ell^2 + \frac{N}{12}\sum_{k\leq M} z_k^4 \\ - \sum_{k<\ell} z_k z_\ell \Big(\sum_{i\leq N} \eta_{i,k}\eta_{i,\ell}\Big).\end{aligned} \tag{1.5}$$

Then, with probability at least $1 - N^{-2}$ (in the variables $\eta_{i,k}$), for each Borel subset A of $\mathbb{R}^M$ we have

$$T^{-1}\overline{G}_1(A) - \frac{1}{N^2} \leq \overline{G}(A) \leq T\overline{G}_1(A) + \frac{1}{N^2} \tag{1.6}$$

for

$$T = \exp\big(LM\sqrt{\alpha}(\log N)^5\big).$$

Even though we are interested in G' rather than $\overline{G}$, we have formulated Theorem 3.1 in terms of $\overline{G}$ because the formulation is cleaner. Of course Theorem 3.1 can be used to relate G' and $\overline{G}_1$ using (1.3). The case where $T \to 1$ is of special interest, because in that case $\overline{G}_1$ and $\overline{G}$ are virtually identical. This occurs as soon as

$$M\sqrt{\alpha}(\log N)^5 \to 0,$$

or, equivalently, $M^3(\log N)^{10}/N \to 0$. In that case, after rescaling, Theorem 1.1 is simply a different formulation of the results of [G-L]. Theorem 1.1 is however of interest even if it is not true that $T \to 1$, but only that $\alpha(\log N)^{10} \to 0$.

Theorem 1.2. *If $M(N) \to \infty, M(N) \le N/(\log N)^{11}$, then for each $\epsilon > 0$ we have*

$$EG_{N,M}(\{\boldsymbol{\sigma}; \sum_{k\le M} m_k^2(\boldsymbol{\sigma}) \notin [(2-\epsilon)\sqrt{\alpha}, (2+\epsilon)\sqrt{\alpha}]\}) \to 0. \tag{1.7}$$

There E denotes of course expectation in the r.v. $\eta_{i,k}$. We can also write (1.7) as

$$EG'_{N,M}(\{\boldsymbol{z}; (2-\epsilon)\sqrt{\alpha} \le \|\boldsymbol{z}\|^2 \le (2+\epsilon)\sqrt{\alpha}\}) \to 1. \tag{1.8}$$

Theorem 1.2 stresses the fact that $\alpha^{1/4}$ is the correct scale to study G'. How does G' look at this scale? Before we state our next result, we must provide motivation. The essential feature of the replica symmetric regime studied in [T2] is that given two generic configurations $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2$ (weighted for Gibbs' measure) the quantity $\sum_{k\le M} m_k(\boldsymbol{\sigma}^1)m_k(\boldsymbol{\sigma}^2)$ is essentially independent of $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2$, and the disorder. It is then natural to ask whether the same is true here (taking in account the proper scale $\alpha^{1/4}$). One should observe first that G' is symmetric around zero (in [T2] "symmetry breaking" terms are introduced to prevent this) so we can at best hope that $|\sum_{k\le M} m_k(\boldsymbol{\sigma}^1)m_k(\boldsymbol{\sigma}^3)|$ is essentially constant. It turns out that this is the case. The physical interpretation is that after rescaling, G' can be seen as a super position of two pure states related by a global symmetry around zero (see [T1], §5 for a similar situation that is also handled rigorously).

Theorem 1.3. *If $M(N) \to \infty, M(N) \le N/(\log N)^{11}$, then for each $\epsilon > 0$, as $N \to \infty$ we have*

$$EG^{\otimes 2}_{N,M}(\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2); |\sum_{k\le M} m_k(\boldsymbol{\sigma}^1)m_k(\boldsymbol{\sigma}^2)| \notin [(1-\epsilon)\sqrt{\alpha}, (1+\epsilon)\sqrt{\alpha}]\}) \to 0. \tag{1.9}$$

In fact, even though it is hard to give a formal statement in this direction, the underlying idea of Theorem 1.3 is that $\overline{G}$ (and hence G') is close to the scaling by a factor $\sqrt{2}(MN)^{-1/4}$ of the Gibbs measure of the spherical version of the Sherrington-Kirkpatrick model at temperature 1/2, a fact that should become clear while reading Sections 4 and 5.

Of course, one is also interested in the sum $\sum_{i\leq N} \sigma_i^1\sigma_i^2$, the behavior of which is described by the following.

Theorem 1.4. *If $M(N)\to\infty, M(N)\leq N/(\log N)^{11}$ then for each $\epsilon>0$, we have*

$$EG_{N,M}^{\otimes 2}(\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2); |\frac{1}{N}\sum_{i\leq N}\sigma_i^1\sigma_i^2| \notin [(1-\epsilon)\sqrt{\alpha},(1+\epsilon)\sqrt{\alpha}]) \to 0. \tag{1.10}$$

Let us now say a word about the proofs. The most striking fact is that, even though one would expect that Theorems 1.1 to 1.3 could be (and, actually, should be) proved using (1.4) only, we will on several occasions heavily use the fact that (1.4) arises from Gibbs measure (1.2). A nice aspect of the proofs is that they contain occurrences of several techniques that have been previously used in more complicated situations; so that the present paper could be used as an introduction to several important ideas about spin glasses.

2. Gaining control.

In this section we prove the basic facts that give us a rough understanding of what happens. These will of course be basic in the proof of the Theorems. The first result asserts that $\alpha^{1/4}$ is the correct scale at which to study $\overline{G}$.

Theorem 2.1. *For some constant L, if $t>0$ and if $Lt\alpha^{1/4}\leq 1$, with probability $\geq 1-\exp(-t^4M)$ we have*

$$\overline{G}(\{\|\boldsymbol{z}\|\geq(1+t)L\alpha^{1/4}\})\leq L\exp(-t^4M). \tag{2.1}$$

This result relies upon a lower bound and an upper bound.

Lemma 2.2. *We have*

$$I =: \int \exp -\psi(\boldsymbol{z})d\boldsymbol{z} \geq \Big(\frac{1}{L\sqrt{\alpha}N}\Big)^{M/2}. \tag{2.2}$$

There and throughout the paper, $d\boldsymbol{z}$ denotes integration with respect to Lebesgue measure λ on $\mathbb{R}^M$. The bound (2.2) holds for *all* values of the numbers $(\eta_{i,k})$, and mimics the proof of Proposition 3.2 of [T1].

Proof of Lemma 2.2. We denote by dU the uniform probability on the orthogonal group S_M of $\mathbb{R}^M$. Since any U in S_M leaves λ invariant, we have

$$\begin{aligned} I &= \int_{\mathbb{R}^M} \int_{S_M} \exp(-\psi(U(\boldsymbol{z})))d\boldsymbol{z}dU \\ &\geq \int_{\mathbb{R}^M} \exp(-\int_{S_M} \psi(U(\boldsymbol{z}))dU)d\boldsymbol{z} \end{aligned}$$

by Jensen's inequality. Now

$$\log \operatorname{ch} y \geq \frac{y^2}{2} - Ly^4$$

so that

$$\begin{aligned} \int \log \operatorname{ch} \boldsymbol{\eta}_i \cdot U(\boldsymbol{z})dU &\geq \frac{1}{2}\int (\boldsymbol{\eta}_i \cdot U(\boldsymbol{z}))^2 dU - L\int (\boldsymbol{\eta}_i \cdot U(\boldsymbol{z}))^4 dU \\ &\geq \frac{\|\boldsymbol{z}\|^2}{2} - L\|\boldsymbol{z}\|^4 \end{aligned}$$

because $\int (\boldsymbol{a} \cdot U(\boldsymbol{z})))^{2p} dU = c_p \|\boldsymbol{a}\|^{2p} \|\boldsymbol{z}\|^{2p}/M$ (and $c_2 = 2$).

Thus, for all $u > 0$,

$$\begin{aligned} I &\geq \int \exp(-LN\|\boldsymbol{z}\|^4)d\boldsymbol{z} \\ &\geq \lambda(\{\|\boldsymbol{z}\| \leq u\}) \exp(-LNu^4) \\ &\geq (\frac{u}{L\sqrt{M}})^M \exp(-LNu^4). \end{aligned}$$

We obtain the result with $u^4 = \alpha = M/N$. □

Lemma 2.3. *If $u\sqrt{\alpha} \leq 1$ we have*

$$P(\forall \boldsymbol{z}, \boldsymbol{z}' \in \mathbb{R}^M, \sum_{i \leq N} (\boldsymbol{\eta}_i \cdot \boldsymbol{z})(\boldsymbol{\eta}_i \cdot \boldsymbol{z}') \leq N\boldsymbol{z} \cdot \boldsymbol{z}' + LNu\sqrt{\alpha}\|\boldsymbol{z}\|\|\boldsymbol{z}'\|) \geq 1 - \exp(-M(L - u^2)). \tag{2.3}$$

Proof. Simple adaptation of the proof of [T1], Lemma 11.3.

Lemma 2.4. *Consider $t > 0$, with $Lt\alpha^{1/4} \leq 1$. Then if the event of Lemma 2.3 occurs for u such that $t = L\sqrt{u}$, we have*

$$J(t) = \int_{\|\boldsymbol{z}\| \geq t\alpha^{1/4}} \exp -\psi(\boldsymbol{z})d\boldsymbol{z} \leq (\frac{L}{\sqrt{\alpha}N})^{M/2} \exp(-\frac{Mt^4}{L}). \tag{2.4}$$

Remark. Even though (2.4) holds for all t, only for $t > L$ can we ensure that the event of Lemma 2.3 occurs with probability close to one.

Proof. It is a elementary fact that the function $x \to \log \operatorname{ch} \sqrt{z}$ is concave on $\mathbb{R}^+$, so that

$$\begin{aligned}\frac{1}{N}\sum_{i\le N}\log \operatorname{ch}|\boldsymbol{\eta}_i\cdot \boldsymbol{z}| &\le \log \operatorname{ch}\sqrt{\frac{1}{N}\sum_{i\le n}(\boldsymbol{\eta}_i\cdot \boldsymbol{z})^2}\\ &\le \log \operatorname{ch}\|\boldsymbol{z}\|(1+Lu\sqrt{\alpha})\end{aligned}$$

under the event of Lemma 2.3. Using that $\log \operatorname{ch} x \le x^2/2 - x^4/L$ if $|x| \le 1$ and $\le ax^2$ for a certain $a < 1/2$ if $x \ge 1$, we see that we have

$$J(t) \le J_1 + J_2$$

where

$$\begin{aligned}J_1(t) &= \int_{\|\boldsymbol{z}\|\ge t\alpha^{1/4}} \exp N(Lu\sqrt{\alpha}\|\boldsymbol{z}\|^2 - \frac{\|\boldsymbol{z}\|^4}{L})d\boldsymbol{z}\\ J_2 &= \int_{\|\boldsymbol{z}\|\ge 1} \exp(-\frac{N}{2}\|\boldsymbol{z}\|^2 + Na\|\boldsymbol{z}\|^2(1+Lu\sqrt{\alpha})^2)d\boldsymbol{z}\end{aligned}$$

If $t = L\sqrt{u}$, we then have

$$\begin{aligned}J_1(t) &\le \int_{\|\boldsymbol{z}\|\ge t\alpha^{1/4}} \exp(-N\frac{\|\boldsymbol{z}\|^4}{L})d\boldsymbol{z}\\ &\le (\frac{L}{N\sqrt{\alpha}})^{M/2}\exp(-Mt^4/L)\end{aligned}$$

by going to polar coordinates (a line of arguments to be more detailed later). Since $t = L\sqrt{u}$, if $Lt\alpha^{1/4} \le 1$, then

$$J_2 \le \int_{\|\boldsymbol{z}\|\ge 1} \exp(-N\frac{\|\boldsymbol{z}\|^2}{L})d\boldsymbol{z} \le \exp(-N/L)$$

as is easily seen. (This is a gaussian integral...) The result follows. □

Proof of Theorem 2.1. It follows from Lemmas 2.2 to 2.4, replacing t by $L(1+t)$ in Lemma 2.4. □

We now start another line of investigation, that finishes with Corollary 2.9. These results are not essential for the paper, but they yield a considerably simpler proof of Theorem 1.2 in the case where $M(N)/\log N \to \infty$. At first reading the reader should certainly content himself with these simpler arguments.

Lemma 2.5. *If $P(\eta_i = 1) = P(\eta_i = -1) = 1/2$ and the r.v. (η_i) are independent, then for $x > 0$ we have*

$$E(1_{\{|m| \geq x\}} \exp \frac{N}{2} m^2) \leq L\sqrt{N} \exp(-\frac{x^4 N}{L}) \tag{2.5}$$

where $m = N^{-1} \sum_{i \leq N} \eta_i$.

Proof. A simple consequence of the Chernov bounds is that for $t \geq 0$,

$$P(|m| \geq t) \leq \exp(-N(\frac{t^2}{2} + \frac{t^4}{L})). \tag{2.6}$$

Next, we use that

$$E(f(|m|) 1_{\{|m| \geq x\}}) = P(|m| \geq x) f(x) + \int_x^\infty f'(t) P(|m| \geq t) dt$$

for $f(x) = \exp \frac{Nx^2}{2}$, and we see that the left-hand side of (2.6) is bounded by

$$\exp -(\frac{Nx^4}{L}) + \int_x^\infty Nt \exp(\frac{-t^4 N}{L}) dt.$$

Now

$$\int_x^\infty Nt \exp \frac{-t^4 N}{L} dt = \sqrt{N} \int_{xN^{1/4}}^\infty v \exp \frac{-v^4}{L} dv \leq L\sqrt{N} \exp(-\frac{x^4 N}{L}). \qquad \square$$

Proposition 2.6. *For $k \leq M$ and $x \geq 0$, we have that*

$$E(G(\{|m_k| \geq x\})) \leq L\sqrt{N} \exp(-\frac{x^4 N}{L}). \tag{2.7}$$

Proof. We use the cavity method "on M". If we denote by $\langle \cdot \rangle_1$ the Gibbs measure associated to the Hamiltonian

$$H_{N,M-1}(\boldsymbol{\sigma}) = -\frac{N}{2} \sum_{k \leq M-1} (m_k(\boldsymbol{\sigma}))^2$$

we then have the identity

$$\langle 1_{\{|m_M| \geq x\}} \rangle = \frac{\langle 1_{\{|m_M| \geq x\}} \exp \frac{N}{2} m_M^2 \rangle_1}{\langle \exp \frac{N}{2} m_M^2 \rangle_1}$$

so that

$$EG(\{|m_M| \geq x\}) \leq E(\langle 1_{\{|m_M| \geq x\}} \exp \frac{N}{2} m_M^2 \rangle_1)$$

and the result from Lemma 2.5 integrating first in the variables $\eta_{i,M}$. $\square$

Comment. It is probable that the factor $\sqrt{N}$ is not necessary in the right hand side of (2.7). If this factor could be removed our simpler approach to Theorem 1.2 would work not only when $M(N)/\log N \to \infty$, but as soon as $M(N) \to \infty$. Removing this factor appears unfortunately to be a difficult problem in itself. Since theorem 1.2 does require $M(N) \to \infty$, it is not so surprising that its proof is easier when there is more "room" and $M(N)$ does not go to infinity too slowly.

Corollary 2.7. *If* $0 \le x \le N$, *we have*

$$EG(\{\sum_{k\le M} m_k^4 \ge \frac{L(x+\sqrt{xM})}{N}\}) \le L\sqrt{N}M\exp(-x) \tag{2.8}$$

Proof. By Proposition 2.6 we have for $u \ge 0$

$$EG(\{\max_{k\le M} m_k^2 \ge LuN^{-1/2}\}) \le L\sqrt{N}M\exp(-u^2). \tag{2.9}$$

By Theorem 2.1, for $t > 0, Lt^4M \le N$, with probability at least $1 - L\exp(-t^4M)$ we have

$$\overline{G}(\{\|\boldsymbol{z}\| \ge (1+t)L\alpha^{1/4}\}) \le L\exp(-t^4M)$$

so that, since $\overline{G} = G' * \gamma$, and $\gamma(\{\|\boldsymbol{z}\| \le L\alpha^{1/4}\}) \ge 1/2$, we have

$$G'(\{\|\boldsymbol{z}\| \ge 2(1+t)L\alpha^{1/4}\}) \le L\exp(-t^4M)$$

and thus

$$EG(\{\sqrt{\sum_{k\ge M} m_k^2} \ge 2(1+t)L\alpha^{1/4}\}) \le L\exp(-t^4M)$$

so that we have

$$EG(\{\sum_{k\le M} m_k^2 \ge L(1+t^2)\sqrt{\frac{M}{N}}\}) \le L\exp(-t^4M).$$

Since

$$\sum_{k\le M} m_k^4 \le (\sum_{k\le M} m_k^2)(\max_{k\le M} m_k^2),$$

we have, using (2.9),

$$EG(\{\sum_{k\le M} m_k^4 \ge L(1+t^2)u\frac{\sqrt{M}}{N}\}) \le L\sqrt{N}M(\exp(-u^2) + \exp(-t^4M)).$$

To conclude we take $u = \sqrt{x}, t^2 = \sqrt{x}/\sqrt{M}$. □

Proposition 2.8. *If* $0 \le x \le 1$ *and* $k \le M$ *we have*

$$E\overline{G}(\{|z_k| \ge x\}) \le L\sqrt{N}\exp(-\frac{Nx^4}{L}) \tag{2.10}$$

This is a result that one would like to prove directly from (1.4); but we don't know how to do this.

Proof. We use that $\overline{G} = G' * \gamma$, so that

$$\begin{aligned}\overline{G}(\{|z_k| \ge x\}) &\le G'(\{|z_k| \ge \frac{x}{2}\}) + \gamma(\{|z_k| \ge \frac{x}{2}\})\\ &\le G(\{|m_k| \ge \frac{x}{2}\}) + \exp(-\frac{Nx^2}{L})\end{aligned}$$

and the result from (2.7). □

Corollary 2.9. *If* $0 \le x \le N$ *then*

$$E\overline{G}(\{\sum_{k\le M} z_k^4 \ge \frac{L(x+\sqrt{xM})}{N}\}) \le LN^{3/2}\exp(-x).$$

Proof. Almost identical to that of Corollary 2.7.

The thrust of Proposition 2.8 is that $|z_k|$ is suitably small for $\overline{G}$. The next result shows that this is also the case for $\boldsymbol{z}\cdot\boldsymbol{\eta}_i$.

Proposition 2.10. *If* $x \ge 0$, *we have*

$$E\overline{G}(\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_i| \ge Lx\alpha^{1/4}\}) \le L\exp(-x). \tag{2.11}$$

Proof. Consider the probability $\overline{G}_0$ on $\mathbb{R}^M$ of density $\exp -\phi_0(\boldsymbol{z})$, where

$$\phi_0(\boldsymbol{z}) = \frac{N\|\boldsymbol{z}\|^2}{2} - \sum_{i\le N-1}\log\operatorname{ch}\boldsymbol{z}\cdot\boldsymbol{\eta}_i$$

so that

$$\overline{G}(\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N| \ge v\}) = \frac{\int \operatorname{ch}\boldsymbol{z}\cdot\boldsymbol{\eta}_N 1_{\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N|\ge v\}}d\overline{G}_0(\boldsymbol{z})}{\int \operatorname{ch}\boldsymbol{z}\cdot\boldsymbol{\eta}_N d\overline{G}_0(\boldsymbol{z})}$$

and thus

$$\begin{aligned}\overline{G}(\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N| \ge v\}) \le \int_{\|\boldsymbol{z}\|\le L(1+t)\alpha^{1/4}} &\operatorname{ch}\boldsymbol{z}\cdot\boldsymbol{\eta}_N 1_{\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N|\ge v\}}d\overline{G}_0(\boldsymbol{z}) \\ &+ \overline{G}(\{\|\boldsymbol{z}\| \ge L(1+t)\alpha^{1/4}\})\end{aligned} \tag{2.12}$$

Now, denoting by E_0 expectation in $\boldsymbol{\eta}_N$, we have

$$\begin{aligned}E_0^2(\operatorname{ch}\boldsymbol{z}\cdot\boldsymbol{\eta}_N 1_{\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N|\ge v\}}) &\le E_0(1_{\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N|\ge v\}})E_0(\operatorname{ch}^2\boldsymbol{z}\cdot\boldsymbol{\eta}_N) \\ &\le \exp(-\frac{v^2}{2\|\boldsymbol{z}\|^2} + 2\|\boldsymbol{z}\|^2).\end{aligned}$$

Since $\boldsymbol{\eta}_N$ is independent of $\overline{G}_0$, taking expectation in (2.12) and applying Theorem 2.1 we see that for $v \ge L(1+t^2)\alpha^{1/2}$ we have, if $t^4M \le N$,

$$E\overline{G}(\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N| \ge v\}) \le \exp(-\frac{v^2}{L(1+t^2)\alpha^{1/2}}) + L\exp(-t^4M). \tag{2.13}$$

To prove (2.11) we can assume $x \ge 1$. We then take $v = Lx\alpha^{1/4}, t = \sqrt{x}$. $\square$

3. Proof of Theorem 1.1.

Proposition 2.10 tells us that to understand G, one has only to consider the values of $\boldsymbol{z}$ where each $|\boldsymbol{z}\cdot\boldsymbol{\eta}_i|$ is rather small. This of course greatly helps in approximating the function ψ of (1.4) using a Taylor expension of $\log\operatorname{ch}$ at zero.

Our next task is to prove a result corresponding to Theorem 2.1 and Proposition 2.10 for $\overline{G}_1$. Thus we will know that certain sets do not matter either for $\overline{G}$ or $\overline{G}_1$, and the task of approximating $\overline{G}_1$ by $\overline{G}$ will be reduced to the easier task of performing this approximation outside these sets.

Proposition 3.1. *For some constant L, and all t with $0 \le Lt\alpha^{1/4} \le 1$, with probability $\ge 1-\exp(-t^4M)$ we have*

$$E\overline{G}_1(\{\|\boldsymbol{z}\| \ge (1+t)L\alpha^{1/4}\}) \le L\exp(-t^4M). \tag{3.1}$$

Proof. It is nearly identical to that of Theorem 2.1.

Lemma 3.2. *If $Mx^4 \le N$ then*

$$E\overline{G}(\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_i| \ge Lx\alpha^{1/4}\}) \le L\exp(-x). \tag{3.2}$$

Proof. It is essentially identical to the proof of Proposition 2.10. We consider the probability $\overline{G}_2$ on $\mathbb{R}^M$ of density $\exp -\psi_2(\boldsymbol{z})$ where

$$\begin{aligned}\psi_2(\boldsymbol{z}) = \frac{N}{2}\sum_{k<\ell} z_k^2 z_\ell^2 + \frac{N}{12}\sum_{k\le M} z_k^4 \\ -\sum_{k\le\ell} z_k z_\ell\Big(\sum_{i\le N-1}\eta_{i,k}\eta_{i,\ell}\Big)\end{aligned}$$

so that

$$\overline{G}_1(\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N| \ge v\}) = \frac{\int \exp(\sum_{k<\ell} z_k z_\ell \eta_{N,k}\eta_{N,\ell}) 1_{\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N|\ge v\}} d\overline{G}_2(\boldsymbol{z})}{\int \exp(\sum_{k<\ell} z_k z_\ell \eta_{N,k}\eta_{N,\ell}) d\overline{G}_2(\boldsymbol{z})}$$

We now observe that

$$\sum_{k<\ell} z_k z_\ell \eta_{N,k}\eta_{N,\ell} = \frac{1}{2}((\boldsymbol{\eta}_N\cdot\boldsymbol{z})^2 - \|\boldsymbol{z}\|^2).$$

A minor complication compared with the case of Proposition 2.10 arises from the fact that this quantity is not always ≥ 0. It is however $\ge -\|\boldsymbol{z}\|^2$, and thus

$$\int \exp(\sum_{k<\ell} z_k z_\ell \eta_{N,k}\eta_{N,\ell}) d\overline{G}_2(z) \ge \frac{1}{e} G_2(\{\|\boldsymbol{z}\| \le 1\})$$

and

$$\overline{G}_1(\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N|\geq v\})\leq \overline{G}_1(\{\|\boldsymbol{z}\|\geq L(1+t)\alpha^{1/4}\})+LA/B$$

for

$$A=\int_{\|\boldsymbol{z}\|\leq L(1+t)\alpha^{1/4}}\exp\frac{1}{2}(\boldsymbol{\eta}_N\cdot\boldsymbol{z})^2 1_{\{|\boldsymbol{z}\cdot\boldsymbol{\eta}_N|\geq v\}}d\overline{G}_1(\boldsymbol{z})$$
$$B=\overline{G}_2(\{\|\boldsymbol{z}\|\leq 1\}).$$

One then observes that if $\|\boldsymbol{z}\|\leq 1/2$ then

$$E\exp(\boldsymbol{\eta}_N\cdot\boldsymbol{z})^2\leq L$$

and that $B\geq 1/2$ with overwhelming probability to conclude as in Proposition 2.10. □

Lemma 3.3. *If $u>0$ and $\boldsymbol{z}\in B_u$, where*

$$B_u=\{\|\boldsymbol{z}\|;\forall i\leq N,|\boldsymbol{z}\cdot\boldsymbol{\eta}_i|\leq u\}\tag{3.3}$$

then

$$|\psi(\boldsymbol{z})-\psi_1(\boldsymbol{z})|\leq \mathrm{I}+\mathrm{II}+\mathrm{III}\tag{3.4}$$

where

$$\mathrm{I}=Lu^4\sum_{i\leq N}(\boldsymbol{z}\cdot\boldsymbol{\eta}_i)^2\tag{3.5}$$

$$\mathrm{II}=\frac{1}{12}|\sum_{i\leq N}(\xi_u(\boldsymbol{\eta}_i\cdot\boldsymbol{z})-E\xi_u(\boldsymbol{\eta}_i\cdot\boldsymbol{z}))|\tag{3.6}$$

$$\mathrm{III}=\frac{1}{12}|\sum_{i\leq N}(E\xi_u(\boldsymbol{\eta}_i\cdot\boldsymbol{z})-E(\boldsymbol{\eta}_i\cdot\boldsymbol{z})^4)|\tag{3.7}$$

for

$$\xi_u(x)=\min(x^4,u^4).$$

Proof. We have

$$\begin{aligned}\psi(\boldsymbol{z})-\psi_1(\boldsymbol{z})&=\frac{N}{2}\|\boldsymbol{z}\|^2-\sum_{i\leq N}\log\operatorname{ch}\boldsymbol{\eta}_i\cdot\boldsymbol{z}\\&-\frac{N}{12}\sum_{k\leq M}z_k^4-\frac{N}{2}\sum_{k<\ell}z_k^2z_\ell^2+\sum_{k<\ell}z_kz_\ell(\sum_{i\leq N}\eta_{i,k}\eta_{i,\ell})\\&=-\frac{1}{12}\sum_{i\leq N}(\boldsymbol{\eta}_i\cdot\boldsymbol{z})^4+\frac{1}{2}\sum_{i\leq N}(\boldsymbol{\eta}_i\cdot\boldsymbol{z})^2-\sum_{i\leq N}\log\operatorname{ch}\boldsymbol{\eta}_i\cdot\boldsymbol{z}\\&+\frac{1}{12}\sum_{i\leq N}(\boldsymbol{\eta}_i\cdot\boldsymbol{z})^4-\frac{1}{12}\sum_{i\leq N}E(\boldsymbol{\eta}_i\cdot\boldsymbol{z})^4.\end{aligned}$$

Now, we observe that

$$|\log \operatorname{ch} x - \frac{x^2}{2} + \frac{1}{12}x^4| \leq Lx^6$$

and that $(\boldsymbol{\eta}_i \cdot \boldsymbol{z})^4 = \xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z})$ for $\boldsymbol{z} \in B_u$. □

Lemma 3.4. *If $w \geq v$, then*

$$P(\sup_{\|z\| \leq v} |\sum_{i \leq N} (\xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}) - E\xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}))| \geq L\sqrt{NM}u^3 w) \leq \exp(-\frac{w^2}{v^2}M).$$

Proof. The key point is that the random variables

$$Y(\boldsymbol{z}) = \sum_{i \leq N} (\xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}) - E\xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}'))$$

satisfy an inequality of the type

$$\|Y(\boldsymbol{z}) - Y(\boldsymbol{z}')\|_{\psi_2} \leq Lu^3\sqrt{N}\|\boldsymbol{z} - \boldsymbol{z}'\| \tag{3.8}$$

where the Orlicz norm $\|\cdot\|_{\psi_2}$ is associated to the function e^{x^2}. To prove this we observe that

$$|\xi_u(x) - \xi_u(y)| \leq Lu^3|x - y|$$

so that

$$|\xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}) - \xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}')| \leq Lu^3|\boldsymbol{\eta}_i \cdot (\boldsymbol{z} - \boldsymbol{z}')|$$

and thus

$$\|\xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}) - \xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}')\|_{\psi_2} \leq Lu^3\|\boldsymbol{z} - \boldsymbol{z}'\|$$

because $\|\boldsymbol{\eta}_i \cdot \boldsymbol{z}\|_{\psi_2} \leq L\|z\|$. Next, if the r.v. $(X_i)_{i \leq N}$ are centered we have

$$\|\sum_{i \leq N} X_i\|_{\psi_2}^2 \leq t \sum \|X_i\|_{\psi_2}^2$$

because then for each t,

$$E \exp tX_i \leq \exp(Lt^2\|X_i\|_{\psi_2}^2).$$

This proves (3.8), or in other words, that the process $(Y(\boldsymbol{z}))$ is subgaussian, with a subgaussian constant $\leq Lu^3\sqrt{N}$. Lemma 3.4 simply expresses what we know about the tails of the supremum of a subgaussian past a multiple of Dudley's entropy integral, see e.g. Theorem 11.2 of [L-T] for a modern formulation. □

Lemma 3.5. *If* $\|\boldsymbol{z}\| \le v$, *we have*

$$|E\xi_u(\boldsymbol{\eta}_i \cdot \boldsymbol{z}) - E(\boldsymbol{\eta}_i \cdot \boldsymbol{z})^4| \le Lv^4 \exp(-\frac{u^2}{4v^2})$$

Proof. The left-hand side is at most

$$\begin{aligned} E((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^4 1_{\{|\boldsymbol{\eta}_i \cdot \boldsymbol{z}| \ge u\}}) &\le E((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^8)^{1/2} P(|\boldsymbol{\eta}_i \cdot \boldsymbol{z}| \ge u)^{1/2} \\ &\le Lv^4 \exp(-\frac{u^2}{4v^2}). \end{aligned}$$

□

Lemma 3.6. *With probability* $\ge 1 - N^{-2}$ *we can find a subset* B *of* $\mathbb{R}^N$ *with*

$$\overline{G}(B) \ge 1 - N^{-3}, \overline{G}_1(B) \ge 1 - N^{-3} \tag{3.9}$$

$$\forall \boldsymbol{z} \in B, |\psi(\boldsymbol{z}) - \psi_1(\boldsymbol{z})| \le LM\sqrt{\alpha}(\log N)^5. \tag{3.10}$$

Proof. We define $u = L\alpha^{1/4} \log N, v = L\alpha^{1/4}(\log N)^{1/4}, w = L\alpha^{1/4}(\log N)^{3/4}$, and

$$B = \{\boldsymbol{z} \in \mathbb{R}^N; \|\boldsymbol{z}\| \le v, \forall i \le N, |\boldsymbol{\eta}_i \cdot \boldsymbol{z}| \le u\}.$$

Theorem 2.1 and Proposition 2.10 show that $\overline{G}(B) \ge 1 - N^{-3}$ with probability $\ge 1 - N^{-3}$ and Proposition 3.1 and Lemma 3.2 show that $\overline{G}_1(B) \ge 1 - N^{-3}$ with probability $\ge 1 - N^{-3}$. This proves (3.9). To prove (3.10), we first observe that by (2.3)

$$\forall \|\boldsymbol{z}\|, \sum_{i \le N} (\boldsymbol{z} \cdot \boldsymbol{\eta}_i)^2 \le LN\|\boldsymbol{z}\|^2$$

with probability $\ge 1 - \exp(-N/L)$.

Thus, the term I of (3.4) is at most

$$Lu^4v^2 \le L\alpha^{3/2}N(\log N)^{9/2} \le LM\sqrt{\alpha}(\log N)^5.$$

To bound the term II of (3.4) we appeal to Lemma 3.4. This lemma shows that with probability at least

$$1 - \exp(-\frac{w^2}{v^2}M) \ge 1 - \exp(-3M\log N) \ge 1 - \frac{1}{N^3}$$

we have

$$\mathrm{II} \le L\sqrt{NM}u^3w = L\alpha\sqrt{NM}(\log N)^5 = LM\sqrt{\alpha}(\log N)^5.$$

To bound the term III of (3.4), we see that by Lemma 3.5 this term is at most

$$\begin{aligned} LNv^4 \exp(-\frac{u^2}{4v^2}) &\le LN\alpha(\log N)^3 \exp(-\log N) \\ &\le L\alpha(\log N)^3 \le LM\sqrt{\alpha}(\log N)^3 \end{aligned}$$

and in fact is of lower order. The lemma is proved. □

Comment. We have made no effort to obtain the smallest possible power of $\log N$ in (3.10).

Proof of Theorem 1.1. Given a Borel set A of $\mathbb{R}^M$, we write

$$\overline{G}(A) \le \overline{G}(A \cap B) + \overline{G}(B^c) \le \overline{G}(A \cap B) + N^{-3},$$

where B is the set of Lemma 3.6. Let

$$I = \int \exp -\psi(\boldsymbol{z}) d\boldsymbol{z}; I_1 = \int \exp -\psi_1(\boldsymbol{z}) d\boldsymbol{z}.$$

Thus, if $T' = \exp(LM\sqrt{\alpha}(\log N)^5)$, using (3.10) we have

$$\begin{aligned} I &\ge \int_B \exp -\psi(\boldsymbol{z}) d\boldsymbol{z} \ge T'^{-1} \int_B \exp -\psi_1(\boldsymbol{z}) d\boldsymbol{z} \\ &= T'^{-1} \overline{G}_1(B) I_1 \ge T'^{-1}(1 - N^{-3}) I_1 \end{aligned} \tag{3.11}$$

Now,

$$\begin{aligned} \overline{G}(A \cap B) = I^{-1} \int_{A \cap B} \exp -\psi(\boldsymbol{z}) d\boldsymbol{z} &\le I^{-1} T' \int_{A \cap B} \exp -\psi_1(\boldsymbol{z}) d\boldsymbol{z}, \\ = I^{-1} T' \overline{G}_1(A \cap B) I_1 &\le (1 - N^{-3})^{-1} T'^2 \overline{G}_1(A) \end{aligned}$$

using (3.11), so that

$$\overline{G}(A) \le (1 - N^{-3})^{-1} T'^2 \overline{G}_1(A) + N^{-3},$$

and, distinguishing whether $T' \le 2$ or $T' \ge 2$ we see that

$$\overline{G}(A) \le T'^3 \overline{G}_1(A) + LN^{-3}.$$

The rest is similar. □

4. Proof of Theorem 1.2.

We fix $\epsilon > 0$ and we set

$$C_\epsilon = \{\boldsymbol{z} \in \mathbb{R}^M; \|\boldsymbol{z}\| \notin [\sqrt{2}\alpha^{1/4}(1 - \epsilon), \sqrt{2}\alpha^{1/4}(1 + \epsilon)]\}.$$

We will prove that for a certain number $a = a(\epsilon) > 0$ depending on ϵ only we have, as $M, N \to \infty, N/M \to \infty$,

$$P(\{\overline{G}_1(C_\epsilon) \le L \exp(-a(\epsilon)M)\}) \to 1. \tag{4.1}$$

Since in Theorem 1.2 we assume $\alpha(\log N)^{10} \to 0$, (1.6) and (4.1) imply Theorem 1.2.

To simplify notation, throughout the paper we set $A_{k\ell} = N^{-1/2} \sum_{i \le N} \eta_{i,k}\eta_{i,\ell}$ if $k \neq \ell$, and $A_{kk} = 0$.

Consider the function

$$\psi_3(\boldsymbol{z}) = \frac{N}{4}\|\boldsymbol{z}\|_2^4 - \sqrt{N}\sum_{k<\ell} z_k z_\ell A_{k\ell}$$

and the probability $\overline{G}_3$ on $\mathbb{R}^M$ of density proportional to $\exp -\psi_3(\boldsymbol{z})$. Thus $\overline{G}_1$ has a density proportional to $\exp \frac{N}{6} \sum_{k \le M} z_k^4$ with respect to $\overline{G}_3$. To prove (4.1) we will prove the following

Proposition 4.1. *There exists $a_1(\epsilon) > 0$ such that*

$$P(\{\overline{G}_3(C_\epsilon) \le L \exp(-a_1(\epsilon)M)\}) \to 1. \tag{4.2}$$

Proposition 4.2. *Given $\eta > 0$, there exists a number $K(\eta)$ such that*

$$P(\{\int (\exp \frac{N}{5} \sum_{k \le M} z_k^4) d\overline{G}_3(\boldsymbol{z}) \le K(\eta) \exp \eta M\}) \to 1. \tag{4.3}$$

The two Propositions imply (4.1) by Hölder's inequality (and since $\exp N \sum_{k \le M} z_k^4 \ge 1$).

We consider the function φ on $\mathbb{R}^+$ given by

$$\varphi(\beta) = \frac{\beta^2}{4} \text{ if } \beta \le 1$$
$$\varphi(\beta) = \beta - \frac{1}{2}\log\beta - \frac{3}{4} \text{ if } \beta \ge 1.$$

We denote by μ_R the uniform probability on the sphere of $\mathbb{R}^M$ of center zero and radius R. Propositions 4.1, 4.2 rely upon the following.

Lemma 4.3. *Given $\eta > 0, L < \infty$, with probability going to 1 as $N \to \infty$, we have, if N/M and N are large enough,*

$$\forall R \le L\alpha^{1/4}, \exp M(\varphi(\frac{R^2\sqrt{N}}{\sqrt{M}}) - \eta) \tag{4.4}$$
$$\le \int \exp(\sqrt{N}\sum_{k<\ell} z_k z_\ell A_{k\ell}) d\mu_R(\boldsymbol{z}) \le \exp M(\varphi(\frac{R^2\sqrt{N}}{\sqrt{M}}) + \eta)$$

Proof. Making the change of variable $z = R/M^{1/4}$, (and setting $\beta = R^2\sqrt{N}/\sqrt{M}$) we see that (4.4) is equivalent to the fact that

$$\tag{4.5} \begin{aligned} &\forall \beta \leq L^2, \exp M(\varphi(\beta) - \eta) \\ &\leq \int \exp(\frac{\beta}{\sqrt{M}} \sum_{k<\ell} z_k z_\ell A_{k\ell}) d\mu_{\sqrt{M}}(\boldsymbol{z}) \leq \exp M(\varphi(\beta) + \eta) \end{aligned}$$

If the quantities $(A_{k\ell})_{k<\ell}$ were i.i.d. $N(0,1)$, (4.5) would essentially be the statement that the free energy density of the spherical version of the Sherrington-Kirkpatrick (SK) model at inverse temperature β is $-\varphi(\beta)/\beta$ [KTJ]. The integral in (4.5) depends only upon the eigenvalues of the symmetric matrix $(\frac{1}{2}A_{k\ell})$ (with $A_{kk} = 0$). In fact more is true: the computation of the free energy density for the spherical SK model depends only upon the fact that the (normalized) eigenvalues (λ_k) of the symmetric random matrix $W = (\frac{1}{2\sqrt{M}} w_{k\ell})$ with $w_{kk} = 0$ and $(w_{k\ell})_{k<\ell}$ i.i.d $N(0,1)$ satisfy Lemma 4.4 below (so that Lemma 4.3 follows from Lemma 4.4). A detailed proof of the upper bound for the SK model can be found in [C], and has inspired much of Section 5 of the present paper.

Lemma 4.4. *As $N, M \to \infty, N/M \to \infty$, the eigenvalues $(\lambda_k)_{k\leq M}$ of the matrix $(\frac{1}{2\sqrt{M}} A_{k\ell})$ satisfy the following property.*

a) For each bounded continuous function f on $\mathbb{R}$, the r.v. $M^{-1} \sum_{k\leq M} f(\lambda_k)$ converges in law to

$$\frac{2}{\pi} \int f(x)\sqrt{1-x^2}dx.$$

b) For all $\epsilon > 0$, $P(\max_{k\leq M} \lambda_k \leq 1+\epsilon) \to 1$.

The proof of a) can be found in [B-Y] and the proof of b) can be found (among other references) in [B-G-P].

Proof of Proposition 4.1. We will make use of polar coordinates; that is, for some constant c_M depending upon M only (the precise value $c_M = 2\pi^{M/2}/\Gamma(M/2)$ is irrelevant here)

$$\int f(\boldsymbol{z})d\boldsymbol{z} = c_M \int_{\mathbb{R}+} R^{M-1} (\int f(\boldsymbol{z}) d\mu_R(\boldsymbol{z})) dR.$$

Thus, under (4.4), we have

$$\tag{4.6} \int \exp -\psi_3(\boldsymbol{z}) d\boldsymbol{z} \geq c_M \int_{R\leq L\alpha^{1/4}} R^{M-1} \exp(M\varphi(\frac{R^2\sqrt{N}}{\sqrt{M}}) - \frac{NR^4}{4} - M\eta) dR$$

and

$$(4.7)\quad \int_{C_\epsilon^c} \exp -\psi_3(\boldsymbol{z})d\boldsymbol{z} \le \int_{R\ge L\alpha^{1/4}} \exp -\psi_3(\boldsymbol{z})d\boldsymbol{z} + c_M \int R^{M-1}\exp(M\varphi(\frac{R^2\sqrt{N}}{\sqrt{M}}) - \frac{NR^4}{4} + M\eta)dR$$

where the last integral is over $R \le L\alpha^{1/4}, |R-\sqrt{2}\alpha^{1/4}| \ge \epsilon\sqrt{2}\alpha^{1/4}$. Making the change of variables $R = \alpha^{1/4}S$, we see that

$$M\log R + M\varphi(\frac{R^2\sqrt{N}}{\sqrt{M}}) - \frac{NR^4}{4} = M(\xi(S) + \log\alpha^{1/4})$$

where the function ξ has its maximum at $S=\sqrt{2}$. (Indeed, for $S\ge 1$, $\xi(S) = -S^4/4 + S^2 - 3/4$; for $S \le 1, \xi(S) = \log S$). Also for L large enough the first term on the right-hand side of (4.7) gives a lower order contribution (by arguments used in the proof of Theorem 1.1). Proposition 4.1 follows easily. □

We turn to the proof of Proposition 4.2. It is unfortunate that this proof is rather delicate. If we however assume that $M/\log N \to \infty$, there is a much easier argument that we give first.

For L large enough, the set

$$B = \{\boldsymbol{z}; \sum_{k\le M} z_k^4 \le y(N) := L(\frac{\log N + \sqrt{M\log N}}{N})\}$$

satisfies $E\overline{G}(B^c) \to 0$ by Corollary 2.9. Also

$$\overline{G}(C_\epsilon \cap B) \le \exp\frac{Ny(N)}{6}\overline{G}_3(C_\epsilon)$$

and $Ny(N)/M \to 0$ (since we assume $M/\log N \to \infty$) so that in that case Theorem 1.2 follows from (4.2). To make the above argument work as soon as $M \to \infty$ (rather than $M/\log N \to \infty$) would require removing the $\sqrt{N}$ in the right-hand side of (2.5).

We now give a (much harder) proof of Proposition 4.2 that works as soon as $M \to \infty$. It relies upon the following.

Lemma 4.5. *Given $L_2 > 0$, given an integer p, given $\eta > 0$, then the following event occurs with probability $\to 1$ as $N, M, N/M \to \infty$.*

Given any subset J of $\{1,\cdots,M\}$ with card $J = p$, *given $t \le L_2\sqrt{\alpha}$, we have*

$$(4.8)\quad \overline{G}_3(\{\sum_{k\in J} z_k^2 \ge t\}) \le \exp(M\eta - \frac{Nt^2}{4}).$$

Before we prove this, we conclude the proof of Proposition 4.2. It should be obvious (after the work of Theorem 1.1) that to prove Proposition 4.2 it suffices to show that

$$P(\{\int_D \exp\frac{N}{5}\sum_{k\le M} z_k^4 d\overline{G}_3(\boldsymbol{z}) \le K(\eta)\exp 10\eta M\}) \to 1 \tag{4.9}$$

where

$$D = \{\boldsymbol{z}; \|\boldsymbol{z}\| \le L_0\alpha^{1/4}\}$$

for a suitably large L_0. We observe that $N\sum_{k\le M} z_k^4 \le L_0^4 M$ on D. To prove (4.9), it suffices to show that, with probability $\to 1$ as $N, M, N/M \to \infty$, we have, for each $u \le \frac{1}{5}L_0^4 M =: L_1 M$ that

$$\overline{G}_3(D\cap\{\frac{N}{5}\sum_{k\le M} z_k^4 \ge u\}) \le \exp(3\eta M - \delta u), \tag{4.10}$$

where $\delta > 1$ is a number ($\delta = 35/32$ works). Consider an integer p and $x > 0$ to be determined later. Consider

$$\begin{aligned} D_1 &= \{\boldsymbol{z}; \exists J\subset\{1,\cdots,M\}, \operatorname{card} J = p, \frac{N}{5}\sum_{k\in J} z_k^4 \ge \frac{7u}{8}\} \\ D_2 &= \{\boldsymbol{z}; \exists J\subset\{1,\cdots,M\}, \operatorname{card} J = p, \forall k\in J, z_k^2 \ge x\} \\ D_3 &= D\backslash(D_1\cup D_2). \end{aligned}$$

Consider $\boldsymbol{z}\in D_3$, and consider $k_1,\cdots,k_p$ such that $|z_{k_1}|,\cdots,|z_{k_p}|$ are the p largest among the M values $|z_1|,\cdots,|z_M|$. Then, if $J = \{k_1,\cdots,k_p\}$ we write

$$\sum_{k\le M} z_k^4 = \sum_{k\in J} z_k^4 + \sum_{k\notin J} z_k^4.$$

Since $\boldsymbol{z}\notin D_1$, we have $\frac{N}{5}\sum_{k\in J} z_k^4 \le \frac{7u}{8}$. Since $\boldsymbol{z}\notin D_2$, by construction of J we have $z_k^2 \le x$ for $k\notin J$, and

$$\sum_{k\notin J} z_k^4 \le x\sum_{k\le M} z_k^2 \le xL_0^2\sqrt{\alpha}.$$

This shows that provided

$$NxL_0^2\sqrt{\alpha} = \frac{u}{8} \tag{4.11}$$

we have

$$D\cap\{\boldsymbol{z}; \frac{N}{5}\sum_{k\le M} z_k^4 > u\} \subset D_1\cup D_2.$$

It follows from Lemma 4.5 that with probability $\to 1$ as $N, M, N/M \to \infty$, we have,

$$\overline{G}_3(D_2) \le M^p \exp(M\eta - \frac{Npx^2}{4}) = M^p \exp(M\eta - \frac{pu^2}{M2^8L_0^4}), \tag{4.12}$$

where we have used (4.11). Now,

$$\frac{pu^2}{2^8ML_0^4} \ge 2u - \frac{2^8L_0^4M}{p}$$

so that if $p = p(\eta)$ is the smallest integer such that

$$\frac{2^8L_0^4}{p} \le \eta \tag{4.13}$$

then (4.12) gives

$$\overline{G}_3(D_2) \le M^p \exp(2M\eta - 2u)$$

To bound $\overline{G}_3(D_1)$, we observe that

$$\sum_{k\in J} z_k^4 \ge \frac{35u}{8N} \Rightarrow \sum_{k\in J} z_k^2 \ge t = \sqrt{\frac{35u}{8N}}$$

and (4.8) then shows that

$$\overline{G}_3(D_1) \le M^p \exp(M\eta - \frac{35u}{32}).$$

Finally we observe that $M^p \exp(2M\eta) \le K(\eta) \exp 3M\eta$. This concludes the proof of Proposition 4.2. $\square$

Proof of Lemma 4.5. We write

$$\left(\int \exp -\psi_3(\boldsymbol{z})d\boldsymbol{z}\right)\overline{G}_3(\{\sum_{k>M-p} z_k^2 > t\}) = \int_{D_4} \exp -\psi_3(\boldsymbol{z})d\boldsymbol{z} = \int_{D_5} f(\boldsymbol{y}_0)d\boldsymbol{y}_0 \tag{4.14}$$

where $\boldsymbol{y}_0 = (z_{M-p+1}, \cdots, z_M)$

$$D_4 = \{\boldsymbol{z} \in \mathbb{R}^M; \sum_{k>M-p} z_k^2 \ge t\}$$
$$D_5 = \{\boldsymbol{y}_0; \sum_{k>M-p} z_k^2 \ge t\}$$

and

$$f(\boldsymbol{y}_0) = \int \exp(-\frac{N}{4}(\|\boldsymbol{y}_1\|^2 + \|\boldsymbol{y}_0\|^2)^2 + \sqrt{N}\sum_{k<\ell} A_{k\ell} z_k z_\ell) d\boldsymbol{y}_1,$$

where the integral is on $\mathbb{R}^{M-p}$, and where $\boldsymbol{y}_1 = (z_1, \cdots, z_{M-p})$. We write

$$\sqrt{N}\sum_{k<\ell} A_{k\ell} z_k z_\ell = U_1 + U_2 + U_3$$

where

$$\begin{aligned} U_1 &= \sqrt{N} \sum_{M-p<k<\ell\le M} A_{k\ell} z_k z_\ell \\ U_2 &= \sqrt{N} \sum_{k\le M-p<\ell\le M} A_{k\ell} z_k z_\ell \\ U_3 &= \sqrt{N} \sum_{k,\ell\le M-p} A_{k\ell} z_k z_\ell . \end{aligned}$$

We observe that

$$\begin{aligned} U_2 &= \sum_{k\le M-p<\ell\le M} \sum_{i\le N} \eta_{i,k}\eta_{i,\ell} z_k z_\ell \\ &= \sum_{i\le N} \sum_{M-p<\ell\le M} \eta_{i,\ell} z_\ell (\sum_{k\le M-p} \eta_{i,k} z_k). \end{aligned}$$

Let us denote by E_0 expectation in the variables $\eta_{i,\ell}$ $(M-p<\ell\le M)$ only. Then, (using that $\operatorname{ch} x \le \exp x^2/2$)

$$E_0 \exp U_2 \le \exp \sum_{i\le N} \sum_{M-p<\ell\le M} \frac{z_\ell^2}{2} (\sum_{k\le M-p} \eta_{i,k} z_k)^2 \tag{4.15}$$

and we see using Lemma 2.3 with $u = \alpha^{-1/2}$ that we have

$$\forall \boldsymbol{y} \in \mathbb{R}^M, \sum_{i\le N} (\sum_{1\le k\le M-p} \eta_{i,k} z_k)^2 \le N\|\boldsymbol{y}_1\|^2 (1 + L\alpha^{1/4}) \tag{4.16}$$

with probability $\ge 1 - L\exp -\sqrt{MN}$. Under (4.16) we have

$$E_0 \exp U_2 \le \exp \frac{N}{2}(1 + L\alpha^{1/4})\|\boldsymbol{y}_0\|^2 \|\boldsymbol{y}_1\|^2 .$$

Given a parameter x, consider the event Ω_x defined as

$$\forall k, \ell > M-p, A_{k\ell} \le x.$$

Then, under (4.16) we have

$$E_0(1_{\Omega_x}\exp(U_1+U_2)) \le \exp(\sqrt{N}x(\sum_{k>M-p}|z_k|)^2+\frac{N}{2}(1+L\alpha^{1/4})\|\boldsymbol{y}_0\|^2\|\boldsymbol{y}_1\|^2)$$

and thus (since $(\sum_{k>M-p}|z_k|)^2 \le p\|\boldsymbol{y}_0\|^2$) we have

$$\begin{aligned}E_0(1_{\Omega_x}f(\boldsymbol{y}_0)) \le \int \exp(&-\frac{N}{4}\|\boldsymbol{y}_1\|^4-\frac{N}{4}\|\boldsymbol{y}_0\|^4+U_3\\&+LN\alpha^{1/4}\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_0\|^2+\sqrt{N}xp\|\boldsymbol{y}_0\|^2)d\boldsymbol{y}_1\end{aligned} \tag{4.17}$$

It should be clear that we are only concerned with the domain $\|\boldsymbol{z}\| \le L\alpha^{1/4}$, so that in (4.17) we can assume

$$\|\boldsymbol{y}_o\|^2, \|\boldsymbol{y}_1\|^2 \le L\alpha^{1/2} \text{ and } N\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_0\|^2 \le NL\alpha = ML$$

If we take $x = M^{1/3}$ (among many other choices) then, $P(\Omega_x) \to 1$ and given η and p, for N/M large (i.e. α small) (4.16) gives

$$E_0(1_{\Omega_x}f(\boldsymbol{y}_0)) \le \exp(\eta M-\frac{N}{4}\|\boldsymbol{y}_0\|^4)\int\exp(-\frac{N}{4}\|\boldsymbol{y}_1\|^4+U_3)d\boldsymbol{y}_1$$

and thus

$$E_0(1_{\Omega_x}\int_{D_5}f(\boldsymbol{y}_0)d\boldsymbol{y}_0) \le I_1I_0 \tag{4.18}$$

where

$$I_1=\int\exp(-\frac{N}{4}\|\boldsymbol{y}_1\|^4+\sqrt{N}\sum_{k<\ell\le M-p}A_{k\ell}z_kz_\ell)d\boldsymbol{y}_1,$$
$$I_0=\exp\eta M\int_{D_5}\exp(-\frac{N}{4}\|\boldsymbol{y}_0\|^4)d\boldsymbol{y}_0.$$

Thus (4.18) shows that

$$\int_{D_5}f(\boldsymbol{y}_0)d\boldsymbol{y}_0 \le \exp M\eta I_1I_0 \tag{4.19}$$

with P_0 probability at least $1-\exp(-M\eta)+P_0(\Omega_x^2))$.

The proof of Proposition 4.1 can be modified in the obvious manner to obtain estimates of the type

$$\frac{c_M}{LM^a}(\alpha e)^{M/4} \le \int\exp-\psi_3(\boldsymbol{z})d\boldsymbol{z} \le Lc_M(\alpha e)^{M/4}M^a$$

where a is a number (observe that $\xi(\sqrt{2}) = 1/4$). Using this lower bound for $\int \exp -\psi_3(\boldsymbol{x})d\boldsymbol{x}$, as well as a similar upper bound for $M-p$ rather than p, we see that with probability $\to 1$ as $N, M, N/M \to \infty$, we have

$$I_1 \le \exp \eta M \int \exp(-\psi_3(\boldsymbol{z}))d\boldsymbol{z}. \tag{4.20}$$

It is elementary to show using polar coordinates that, for N large

$$\int_{D_5} \exp(-\frac{N}{4}\|\boldsymbol{y}_0\|^4)d\boldsymbol{y}_0 \le \exp(-\frac{Nt^4}{4}),$$

using the fact that we assume $t \le L_2\sqrt{\alpha} \le 1$.

Recalling (4.14), (4.18) we see that under (4.19), given t, we have

$$\overline{G}_3(\{\sum_{k>M-p} z_k^2 \ge t\}) \le \exp(2M\eta - \frac{Nt^2}{4}) \tag{4.21}$$

with P_0-probability $\ge 1 - P_0(\Omega_x) - \exp(-M\eta)$. On the other hand, to ensure (4.21) for all t (with $t \le L_2\sqrt{\alpha}$) it suffices (replacing $2M\eta$ by $3M\eta$), to ensure it for the values of t with $Nt^2/4$ integer and there are about M of these. The only point remaining is to show that we can do the above not only when $J = (M, M-1, \cdots, M-p)$, but simultaneously for each subset J of $\{1, \cdots, M\}$ of cardinality p. There are fewer than M^p of these; the only problem could be with (4.19), as we have not proved that the probability where this fails is exponentially small. The validity of (4.19) depends only upon the eigenvalues matrices obtained from $(A_{k\ell})$ by removing p rows and the corresponding columns. These matrices are close to each other in operator norm, so their eigenvalues do not differ much. □

5. Proof of Theorem 1.3.

After having proved Theorem 1.2 through Propositions 4.1 and 4.2, it is quite natural to try to prove Theorem 1.3 by the same method, namely by proving that, given $\epsilon > 0$, there exists $a_2(\epsilon) > 0$ that

$$P(\{\overline{G}_3^{\otimes 2}(C'_\epsilon) \le L \exp -a_2(\epsilon)M\}) \to 1 \tag{5.1}$$

for

$$C'_\epsilon = \{(\boldsymbol{z}^1, \boldsymbol{z}^2) \in \mathbb{R}^{2M}; |\boldsymbol{z}^1 \cdot \boldsymbol{z}^2| \notin [(1-\epsilon)\sqrt{\alpha}, (1+\epsilon)\sqrt{\alpha}]\}.$$

We will prove “half” of this, namely

Proposition 5.1. *Given $\epsilon > 0$, there is $a_2(\epsilon) > 0$ such that*

$$P(\{\overline{G}_3^{\otimes 2}(C''_\epsilon) \leq L \exp -a_2(\epsilon)M\}) \to 1 \tag{5.2}$$

for

$$C''_\epsilon = \{(\boldsymbol{z}^1, \boldsymbol{z}^2) \in \mathbb{R}^{2M}; |\boldsymbol{z}^1 \cdot \boldsymbol{z}^2| \geq (1+\epsilon)\sqrt{\alpha}\}. \tag{5.3}$$

We have not succeeded in proving "the other half" of (5.1), and it is in fact not clear at all to us that it is true. The difficulty has its roots in the spherical version of the SK model, a model that is more canonical than the one considered here, so it seems worthwhile to explain a bit in detail what happens. We first recall what is the spherical version of the SK model. Given an i.i.d $N(0,1)$ sequence $(g_{k\ell})_{k<\ell}$, and the sphere S_M of $\mathbb{R}^M$ radius $\sqrt{M}$, for $\boldsymbol{z} \in S_M$ consider

$$H_M(\boldsymbol{z}) = -\frac{1}{\sqrt{M}} \sum_{k<\ell} g_{k\ell} z_k z'_\ell \tag{5.4}$$

and, the probability $SK_{M,\beta}$ on S_M of density proportional to $\exp -\beta H_M(\boldsymbol{z})$ with respect to the uniform measure on S_M. (Of course the physicists "know" everything about this model). We have the following rigorous results.

Proposition 5.2. *If $\beta > 0$, for each $\epsilon > 0$ we have*

$$P(\{SK_{M,\beta}^{\otimes 2}(\{(\boldsymbol{z}^1, \boldsymbol{z}^2) \in \mathbb{R}^{2M}; |\boldsymbol{z}^1 \cdot \boldsymbol{z}^2| \geq (1 - \frac{1}{\beta} + \epsilon)M\}) \\ \leq \exp -z_3(\epsilon)M\}) \to 1 \tag{5.5}$$

$$\lim_{M\to\infty} ESK_{M,\beta}^{\otimes 2}(\frac{\boldsymbol{z}^1 \cdot \boldsymbol{z}^2}{M})^2 = (1 - \frac{1}{\beta})^2 \tag{5.6}$$

Combining (5.5) and (5.6), we have

$$ESK_{M,\beta}^{\otimes 2}(||\frac{\boldsymbol{z}^1 \cdot \boldsymbol{z}^2}{N}| - (1 - \frac{1}{\beta})||) \to 0 \tag{5.7}$$

Problem 5.3. Given $\epsilon > 0$, is it true that with probability $\to 1$ we have

$$SK_{M,\beta}^{\otimes 2}\{(\boldsymbol{z}^1, \boldsymbol{z}^2); |\boldsymbol{z}^1 \cdot \boldsymbol{z}^2| \leq 1 - \frac{1}{\beta} - \epsilon\} < \exp -a(\epsilon)M \tag{5.8}$$

where $a(\epsilon) > 0$?

The nature of (5.5) and (5.8) are rather different. We will prove (5.5) by an adaptation of Comet's argument [C], and (5.5) depends only upon the properties of Lemma 4.4, that is it remains valid if we replace (5.4) by

$$H_M(z) = -\sum_{\ell \leq M} z_\ell^2 \lambda_{\ell,M} \tag{5.9}$$

where

$$\lim_{M\to\infty} \max_{\ell\le M} \lambda_{\ell,M} = 1 \tag{5.10}$$

$$\frac{1}{M}\sum_{\ell\le M} \delta_{\lambda_{\ell,M}} \overset{\text{weakly}}{\longrightarrow} \frac{2}{\pi}\sqrt{1-x^2}dx \tag{5.11}$$

On the other hand, it is not true that (5.8) depends only upon (5.10), (5.11). For example, one can show that if the numbers $\lambda_{\ell,M}$ occur with multiplicity $\to\infty$, then for the corresponding measure G_M one has

$$\int (\frac{\boldsymbol{z}^1\cdot\boldsymbol{z}^2}{M})^2 dG_M(\boldsymbol{z}^1)dG_M(\boldsymbol{z}^2) \to 0.$$

To circumvent our inability to prove (5.8), we will observe that (5.2) together with Proposition 4.2 and Theorem 1.1 imply that

$$P(\{\overline{G}^{\otimes 2}(C''_\epsilon) \le L\exp(-a_4(\epsilon)M)\}) \to 1. \tag{5.12}$$

Using Theorem 2.1, it is easy to see that one can replace $\overline{G}$ by G' in this statement, so that

$$\begin{aligned} P(\{G^{\otimes 2}_{N,M}(\{\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\};|\sum_{k\le M} m_k(\boldsymbol{\sigma}^1)m_k(\boldsymbol{\sigma}^2)| \ge (1+\epsilon)\sqrt{\alpha}\}) \\ \le \exp(-a_5(\epsilon)M)\}| \to 1. \end{aligned} \tag{5.13}$$

We will then prove by completely different arguments that

$$\frac{N}{M}E\langle(\sum_{k\le M} m_k(\boldsymbol{\sigma}^1)m_k(\boldsymbol{\sigma}^2))^2\rangle \to 1, \tag{5.14}$$

completing the proof of Theorem 1.3.

Proposition 5.2 will be deduced from the following.

Proposition 5.4. *Consider on S_M the random probability $W_\beta = W_{M,N,\beta}$ of density*

$$\exp\frac{\beta}{\sqrt{M}}\sum_{k<\ell} z_k z_\ell A_{k\ell} \tag{5.15}$$

with respect to the uniform probability on S_M. Then, for each $\epsilon>0$, there exists $a(\epsilon)>0$ such that

$$\begin{aligned} P(\{\forall\beta,\beta', 1\le\beta,\beta'\le 3; W_\beta\otimes W_{\beta'}(\{(\boldsymbol{z}^1,\boldsymbol{z}^2);|\boldsymbol{z}^1\cdot\boldsymbol{z}^2|\ge\epsilon+c(\beta,\beta') \\ \le\exp(-a(\epsilon)M)\})\to 0 \end{aligned} \tag{5.16}$$

where

$$c(\beta,\beta') = \frac{\beta+\beta'}{2\sqrt{\beta\beta'}} - \frac{1}{\sqrt{\beta\beta'}}.$$

In this statement, the number 3 can be replaced by any other. Since the argument depends only on the properties (5.10), (5.11) of the eigenvalues of the matrix $(\frac{1}{2\sqrt{M}}A_{k\ell})$ it would also hold if we had $g_{k\ell}$ instead of $A_{k\ell}$, and the case $\beta = \beta'$ implies (5.5).

Before we prove Proposition 5.4, we show why it implies Proposition 5.2. Using polar coordinates as in (4.5), we see that

$$\int f(\boldsymbol{z})d\overline{G}_3(\boldsymbol{z}) = c_M \int_{\mathbb{R}+} R^{M-1} \int f(\boldsymbol{z}) \exp(\sqrt{N}\sum_{k<\ell} A_{k\ell}z_k z_\ell)d\mu_R(\boldsymbol{z}) \tag{5.17}$$

and making the change of variables $\boldsymbol{z} = \boldsymbol{y}R/M^{1/4}$ the inner integral is

$$\Big(\int \exp\sqrt{N}\sum_{k<\ell} A_{k\ell}z_k z_\ell)d\mu_R(\boldsymbol{z})\Big) \int f(\frac{R\boldsymbol{y}}{M^{1/4}})dW_{R^2/\sqrt{\alpha}}(\boldsymbol{y}).$$

The content of Theorem 1.2 is that (if, say, f is bounded) only the values of R close to $\sqrt{2}\alpha^{1/4}$ have to be considered in (5.17). Proposition 5.2 follows from Proposition 5.4 for values of β,β' close to 2. The easy details are left to the reader.

To prove Proposition 5.4 we will prove the following .

Proposition 5.5. *Given $\eta > 0$, with probability $\to 1$ as $N, M, N/M \to \infty$, for $1 \le \beta,\beta' \le 3, |t| \le 1$, we have*

$$J := \int \exp t\boldsymbol{z}^1\cdot\boldsymbol{z}^2 dW_\beta(\boldsymbol{z}^1)dW_{\beta'}(\boldsymbol{z}^2) \le \exp M(\eta + \Psi_{\beta,\beta'}(t)), \tag{5.18}$$

where

$$\Psi_{\beta,\beta'}(0) = 0, \tag{5.19}$$

$$\lim_{t\to 0^+} \Psi_{\beta,\beta'}(t)/t = c(\beta,\beta') \tag{5.20}$$

$$\lim_{t\to 0^-} \Psi_{\beta,\beta'}(t)/t = -c(\beta,\beta'). \tag{5.21}$$

To deduce Proposition 5.5 from Proposition 5.4, we proceed as follows. Given $\epsilon > 0$, we choose $t_0 > 0$ such that

$$\Psi_{\beta,\beta'}(t_0) \le t_0(\frac{\epsilon}{3} + c(\beta,\beta')).$$

We use (5.15) with $t = t_0, \eta = \epsilon t_0/3$. Use of Chebishev inequality then show that

$$W_\beta \otimes W_{\beta'}(\{\frac{\boldsymbol{z}^1 \cdot \boldsymbol{z}^2}{M} \geq \epsilon + c(\beta, \beta')\}) \leq \exp -\frac{t_0 \epsilon M}{3},$$

and the other part of (5.16) is similar.

We will prove (5.18) with

$$\Psi_{\beta,\beta'}(t) = |t| \frac{\beta + \beta'}{2\sqrt{\beta\beta'}} - \frac{1}{2}\Phi(1 + \frac{|t|}{\sqrt{\beta\beta'}}), \tag{5.22}$$

where, for $u \geq 1$,

$$\Phi(u) = u(u - \sqrt{u^2 - 1}) + \log(u + \sqrt{u^2 - 1}) - 1.$$

It is elementary to see (using a Taylor expansion of order 2 of $\log(1+x)$...) that

$$\Phi(1 + x) = 2x + o(x) \tag{5.23}$$

where $\lim\limits_{x \to 0^+} o(x)/x = 0$ so that (5.20) and (5.21) follow from (5.22) and (5.23).

The occurrence of $\Phi(u)$ is from the magic formula

$$\frac{2}{\pi} \int \log(u - x)\sqrt{1 - x^2} du = \Phi(u) + \frac{1}{2} - \log 2 \tag{5.24}$$

for $u > 1$.

Proof of (5.18). We consider $s, s' > 0$, and we will always assume that

$$\frac{s'}{\beta'} = \frac{s}{\beta} > 1. \tag{5.25}$$

Consider the integral

$$\begin{aligned} I = \int_{\mathbb{R}^M \times \mathbb{R}^M} \exp\Big[& \frac{\beta}{\sqrt{M}} \sum_{k<\ell} A_{k\ell} z_k^1 z_\ell^1 + \frac{\beta'}{\sqrt{M}} \sum_{k<\ell} A_{k\ell} z_k^2 z_\ell^2 \\ & - s\|\boldsymbol{z}^1\|^2 - s'\|\boldsymbol{z}^2\|^2 + t\boldsymbol{z}^1 \cdot \boldsymbol{z}^2\Big] d\boldsymbol{z}^1 d\boldsymbol{z}^2. \end{aligned} \tag{5.26}$$

By rational invariance of Gaussian measure we have

$$I = \prod_k I_k$$

where

$$I_k = \int_{\mathbb{R}^2} \exp((\beta\lambda_k - s)x^2 + (\beta'\lambda_k - s')y^2 + txy) dx dy$$

as is seen by diagonalization of the quadratic form $\sum_{k,\ell} \frac{1}{2\sqrt{M}} A_{k\ell} z_k z_\ell$ in an orthogonal basis.

The quadratic forms

$$Q_k(x,y) = (s - \beta\lambda_k)x^2 + (s' - \beta'\lambda_k)y^2 + txy$$

are positive definite provided

$$(s - \beta\lambda_k)(s' - \beta'\lambda_k) - \frac{t^2}{4} > 0$$

and using (5.25), this amounts to

$$\frac{s}{\beta} > \lambda_k + \frac{|t|}{2\sqrt{\beta\beta'}}. \tag{5.27}$$

For a positive definite quadratic Q form on $\mathbb{R}^p$, we have

$$\int_{\mathbb{R}^p} \exp Q(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} = \pi^{p/2} \frac{1}{\sqrt{\det Q}}$$

(as follows from diagonalization and the case $p = 1$).

In the case of I_k, we have $p = 2$, and

$$\begin{aligned}\det Q_k &= (s - \beta\lambda_k)(s' - \beta'\lambda_k) - \frac{t^2}{4} \\ &= \beta\beta'(\frac{s}{\beta} - \frac{|t|}{2\sqrt{\beta\beta'}} - \lambda_k)(\frac{s}{\beta} + \frac{|t|}{2\sqrt{\beta\beta'}} - \lambda_k)\end{aligned}$$

so that

$$I_k = \frac{\pi}{\sqrt{\beta\beta'}} \exp\big[-\frac{1}{2}\log(\frac{s}{\beta} - \frac{|t|}{2\sqrt{\beta\beta'}} - \lambda_k) - \frac{1}{2}\log(\frac{s}{\beta} + \frac{|t|}{2\sqrt{\beta\beta'}} - \lambda_k)\big]$$

Thus

$$I = \Big(\frac{\pi}{\sqrt{\beta\beta'}}\Big)^M \exp \sum_{k \le M} \big[-\frac{1}{2}\log\big(\frac{s}{\beta} - \frac{|t|}{2\sqrt{\beta\beta'}} - \lambda_k\big) - \frac{1}{2}\log\big(\frac{s}{\beta} + \frac{|t|}{2\sqrt{\beta\beta'}} - \lambda_k\big)\big].$$

By (5.11), (5.10), (5.24), given $\eta > 0$, and s with $s/\beta > 1 + |t|/2\sqrt{\beta\beta'}$, (5.27) holds for M large enough, and

$$I \le \Big(\frac{\pi}{\sqrt{\beta\beta'}}\Big)^M \exp M\big[\eta - \frac{1}{2}\Phi\big(\frac{s}{\beta} + \frac{|t|}{2\sqrt{\beta\beta'}}\big) - \frac{1}{2} + \log 2\big]. \tag{5.28}$$

If we denote by $I(R,R')$ the integral defined as in (5.26) but with $d\mu_R(\boldsymbol{z}^1)d\mu_{R'}(\boldsymbol{z}^2)$ instead of $d\boldsymbol{z}^1 d\boldsymbol{z}^2$, given $\eta > 0$, there is $\xi > 0$ such that

$$\begin{aligned} &|R - \sqrt{M}| \le \xi\sqrt{M}; |R' - \sqrt{M}| \le \xi\sqrt{M} \\ &\Rightarrow I(R,R') \ge I(\sqrt{M},\sqrt{M}) \exp -\eta M \end{aligned} \tag{5.29}$$

(let us recall that β, β', s, t remain bounded). Thus we have

$$\begin{aligned} I &\ge I(\sqrt{M},\sqrt{M})\lambda(\{\boldsymbol{z}; |\|\boldsymbol{z}\| - \sqrt{M}| \le \xi\sqrt{M}\})^2 \exp -\eta M \\ &\ge I(\sqrt{M},\sqrt{M}) \exp(-\eta M) M^M((1+\xi)^M - (1-\xi)^M)^2 a_M^2 \end{aligned} \tag{5.30}$$

where a_M is the volume of the sphere of radius 1 in $\mathbb{R}^M$, so that, using Stirling's formula

$$a_M = \frac{\pi^{M/2}}{\Gamma(1+M/2)M^{M/2}} \ge \frac{1}{L\sqrt{M}}\Big(\frac{2\pi e}{M^2}\Big)^{M/2}. \tag{5.31}$$

Comparing (5.28) and (5.30) we get (if $M \ge M(\eta)$)

$$\begin{aligned} I(\sqrt{M},\sqrt{M}) \le L\Big(\frac{1}{\beta\beta'}\Big)^{M/2} \exp M\Big[3\eta - \frac{1}{2}\Phi\Big(\frac{s}{\beta} - \frac{|t|}{2\sqrt{\beta\beta'}}\Big) \\ - \frac{1}{2}\Phi\Big(\frac{s}{\beta} + \frac{|t|}{2\sqrt{\beta\beta'}}\Big) - \frac{3}{2}\Big]. \end{aligned} \tag{5.32}$$

On the other hand, the lower bound of (4.5) shows that

$$I(\sqrt{M},\sqrt{M}) \ge \exp(-(s+s')M - 2\eta M) \exp M\Big(\beta + \beta' - \frac{1}{2\log\beta} - \frac{1}{2\log\beta'} - \frac{3}{2}\Big)J$$

and comparison with (5.32) yields

$$J \le \exp \eta M \exp M\Big(s + s' - \beta - \beta' - \frac{1}{2}\Phi\Big(\frac{s}{\beta} - \frac{|t|}{2\sqrt{\beta\beta'}}\Big) - \frac{1}{2}\Phi\Big(\frac{s}{\beta} + \frac{|t|}{2\sqrt{\beta\beta'}}\Big)\Big). \tag{5.33}$$

Since $\lim_{x\to 1}\Phi(x) = 0$, given η we can choose s satisfying (5.27), close enough to $\beta + \beta|t|/2\sqrt{\beta\beta'}$ that

$$\begin{aligned} s + s' - \beta - \beta' - 2 - \frac{1}{2}\Phi\Big(\frac{s}{\beta} - \frac{|t|}{2\sqrt{\beta\beta'}}\Big) - \frac{1}{2}\Phi\Big(\frac{s}{\beta} + \frac{|t|}{2\sqrt{\beta\beta'}}\Big) \\ \le \eta + |t|\Big(\frac{\beta+\beta'}{2\sqrt{\beta\beta'}}\Big) - \frac{1}{2}\Phi\Big(1 + \frac{|t|}{\sqrt{\beta\beta'}}\Big) \end{aligned} \tag{5.34}$$

so that

$$J \le \exp M\Big[5\eta + |t|\frac{\beta+\beta'}{2\sqrt{\beta\beta'}} - \frac{1}{2}\Phi\Big(1 + \frac{|t|}{\sqrt{\beta\beta'}}\Big)\Big]. \tag{5.35}$$

It should be obvious from the proof that given $\eta > 0$, with probability $\to 1$ as $N \to \infty$ this holds for $\beta, \beta', |t|$ bounded. $\square$

We now turn to the proof of (5.14). We know of two rather different ways to proceed. The first method is in the spirit of the previous arguments. It is to show rather than (5.14) that

$$\frac{N}{M} E \int (\frac{\boldsymbol{z}^1 \cdot \boldsymbol{z}^2}{M})^2 d\overline{G}_3(\boldsymbol{z}^1) d\overline{G}_z(\boldsymbol{z}^2) \to 1 \tag{5.36}$$

This can be done using an argument of [ALR]. We know how to compute

$$\xi(\beta) = \lim \frac{1}{M} E \log \Big(\int \exp(-\frac{M}{4}\|\boldsymbol{z}\|_2^2 + \frac{M}{6} \sum_{k \le M} z_k^4 + \frac{\beta}{\sqrt{M}} \sum_{k<\ell} A_{k\ell} z_k z_\ell) d\boldsymbol{z} \Big)$$

which is a convex function of β, so we know how to compute the (limit of the) derivative of the right-hand side in β, at $\beta = 1$. Integration by parts then brings the relation (5.36). (A similar approach in the case of the spherical SK model yields (5.6)). The integrations by parts would be easy if $(A_{k\ell})_{k<\ell}$ were independent gaussian; but since they are not, one has instead to use "approximate integration by parts" in the variables $\eta_{i,k}$ and this is rather tedious. The other approach (the one we choose) is to prove (5.14) using the tools developed by the author to use the cavity method for the Hopfield model. Given these tools, this method is simpler than the previous one, and the tools seem anyway needed to prove Theorem 1.4.

We consider the Hamiltonian

$$H'_{N,M}(\boldsymbol{\sigma}, \sigma_{N+1}) = -\frac{1}{2N} \sum_{k \le 1}^{M} \Big(\sum_{i \le N} \eta_{i,k} \sigma_i + \eta_k \sigma_{N+1} \Big)^2 \tag{5.37}$$

where (η_k) is a fresh independent sequence, $P(\eta_k = \pm 1) = 1/2$. We will denote by $\langle\ \rangle'$ integration with respect to the Gibbs measure associated with (5.37) (at temperature 1). The idea is that computation $E\langle V \rangle'$ in function of quantities involving only $\langle\ \rangle$ in several different ways for well chosen values of U yields interesting relations.

Proposition 5.6. *If f is a (possibly random) function on Σ_N^2, then*

$$E\langle f \rangle' = E\langle f \rangle + E\langle f \boldsymbol{m}^1 \cdot \boldsymbol{m}^2 \rangle + A \tag{5.38}$$

$$E\langle f \sigma_{N+1}^1 \sigma_{N+2}^2 \rangle' = E\langle f \boldsymbol{m}^1 \cdot \boldsymbol{m}^2 \rangle - \frac{1}{2} E\langle f \sum_{\ell=1,2} a_\ell \rangle + A \tag{5.39}$$

where

$$\boldsymbol{m}^1 \cdot \boldsymbol{m}^2 = \sum_{k \le M} m_k(\boldsymbol{\sigma}^1) m_k(\boldsymbol{\sigma}^2); a_\ell = \boldsymbol{m}^\ell \cdot \boldsymbol{m}^\ell - \langle \boldsymbol{m}^\ell \cdot \boldsymbol{m}^\ell \rangle$$

and

$$|A| \le E\langle |f| \big(\sum_{\ell,\ell' \le 3} (\boldsymbol{m}^\ell \cdot \boldsymbol{m}^{\ell'})^2 + \sum_{\ell \le 3} a_\ell^2 \big) \rangle. \tag{5.40}$$

In this statement, f is identified to a function on Σ_{N+1}^2; and in (5.39), the integral is over a 3-replica. A result of the same nature as Proposition 5.6 is proved in [T2], Theorem 2.6, Proposition 5.7. The present case is much simpler. With the notation of [T2], we have $\beta = 1, \boldsymbol{b} = \langle \boldsymbol{m} \rangle = 0$, so that $Y = 0$, and there is no need to consider the terms c_ℓ. The reader who likes to really understand what happens should of course carry out the proof of Proposition 5.6.

Proposition 5.7. *Consider a function f on Σ_{N+1}, and assume that f does not depend upon $(\eta_k)_{k \le M}$. Then*

$$E\eta_k \langle f \rangle' = E\langle m_k \sigma_{N+1} f \rangle' - E\langle m_k \sigma_{N+1} \rangle' \langle f \rangle' + S \tag{5.41}$$

where

$$E|S| \le E\langle Av |f| |m_k|^3 \rangle.$$

There $m_k = N^{-1} \sum_{i \le N} \sigma_i \eta_{i,k}$, Av is average over the values of $\sigma_{N+1} = \pm 1$. Proposition 5.7 is a special case of Proposition 2.12 of [T2] (an elementary approximate integration by parts).

Proof of (5.14). Consider

$$U = \sum_{k \le M} u_k^2$$

where

$$u_k = \frac{1}{N+1} \sum_{i \le N+1} \eta_{i,k} \sigma_i,$$

where we set $\eta_{N+1,k} = \eta_k$. Using symmetry between sites we have (setting $N' = N+1$)

$$\begin{aligned} E\langle U \rangle' &= \sum_{k \le M} E\langle \eta_k \sigma_{N+1} u_k \rangle' \\ &= \frac{M}{N'} + \frac{N}{N'} \sum_{k \le M} E\langle \eta_k \sigma_{N+1} m_k \rangle'. \end{aligned} \tag{5.42}$$

Now m_k does not depend upon η_k so that we can appeal to (5.40) to get

$$\begin{aligned} &\Big| \sum_{k \le M} E\eta_k \langle \sigma_{N+1} m_k \rangle' - \big(E\langle \sum_{\ell \le k} m_k^2 \rangle' - E\langle \boldsymbol{m}^1 \cdot \boldsymbol{m}^2 \sigma_{N+1}^1 \sigma_{N+1}^2 \rangle' \big) \Big| \\ &\qquad \le LE\langle \sum_k m_k^4 \rangle. \end{aligned} \tag{5.43}$$

There we use replicas to write

$$\big(\langle\sigma_{N+1}m_k\rangle'\big)^2 = \langle m_k(\boldsymbol{\sigma}^1)m_k(\boldsymbol{\sigma}^2)\sigma^1_{N+1}\sigma^2_{N+1}\rangle'.$$

We now appeal to (5.39) to get

$$|E\langle \boldsymbol{m}^1\cdot\boldsymbol{m}^2\sigma^1_{N+1}\sigma^2_{N+1}\rangle' - E\langle(\boldsymbol{m}^1\cdot\boldsymbol{m}^2)^2\rangle| \le \frac{1}{2}E\langle|\boldsymbol{m}^1\cdot\boldsymbol{m}^2|\sum_{\ell=1,2}|a_\ell|\rangle + A \tag{5.44}$$

where

$$A \le E\langle|\boldsymbol{m}^1\cdot\boldsymbol{m}^2|\sum(\boldsymbol{m}^\ell\cdot\boldsymbol{m}^{\ell'})^2 + \sum a_\ell^2)\rangle.$$

It should be obvious from Theorem 2.1 that $A = o(\alpha)$. (That is, $A/\alpha \to 0$ as $N, M, N/M \to \infty$). Use of Cauchy-Schwarz and independence of replicas show that

$$\begin{aligned}\langle|\boldsymbol{m}^1\cdot\boldsymbol{m}^2||a_\ell|\rangle &\le \langle(\boldsymbol{m}^1\cdot\boldsymbol{m}^2)^2\rangle^{1/2}\langle(a_\ell)^2\rangle^{1/2}\\ &\le \langle\|\boldsymbol{m}\|^2\rangle\langle(a_\ell)^2\rangle^{1/2}.\end{aligned}$$

The uniform integrability control provided by Theorem 2.1 together with Theorem 1.2 imply that this is also $o(\alpha)$. It follows from Corollary 2.7 that $E\langle\sum m_k^4\rangle = o(\alpha)$ if $M/\log N \to \infty$; but this is true as soon as $M \to \infty$, as follows from (4.3) and Theorem 2.1 (the relevance of which being that $\sum m_k^4 \le (\sum m_k^2)^2$). Combining these estimates, we have shown that

$$E\langle U\rangle' = \frac{M}{N} + E\langle\sum_{k\le M} m_k^2\rangle - E\langle(\boldsymbol{m}^1\cdot\boldsymbol{m}^2)^2\rangle + o(\alpha). \tag{5.45}$$

We now prove that

$$E\langle U\rangle' = E\langle\sum_{k\le M} m_k^2\rangle + o(\alpha) \tag{5.46}$$

which proves (5.15) when combined with (5.45). To do this, we appeal again to the symmetry between sites to write, as in (5.42)

$$\begin{aligned}E\langle U'\rangle' &= \sum_{k\le M} E\langle m_k u_k\rangle'\\ &= \frac{1}{N'}\sum_{k\le M} E\eta_k\langle m_k\sigma_{N+1}\rangle' + \frac{N}{N'}E\langle\sum_{k\le M} m_k^2\rangle'.\end{aligned}$$

Use of (5.43) on the first term, and of (5.38) on the last term easily imply (5.46) and hence (5.15).

Proof of Theorem 1.4. We prove that

$$\lim \frac{N}{M} E\langle(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)^2\rangle = 1 \tag{5.47}$$

$$\lim (\frac{N}{M})^2 E\langle(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)^4\rangle = 1 \tag{5.48}$$

This is easy, but long, so we indicate only how to prove (5.47). The proof relies upon the relations

$$E\langle(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)^2\rangle \simeq E\langle(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)^2\rangle' \tag{5.49}$$

$$E\langle(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)^2\rangle' \simeq E\langle(\frac{1}{N'}\sum_{i\le N+1}\sigma_i^1\sigma_i^2)^2\rangle' \tag{5.50}$$

$$E\langle(\frac{1}{N'}\sum_{i\le N+1}\sigma_i^1\sigma_i^2)^2\rangle' = \frac{1}{N'} + E\langle\sigma_{N+1}^1\sigma_{N+1}^2(\frac{1}{N'}\sum_{i\le N}\sigma_i^1\sigma_i^2)\rangle' \tag{5.51}$$

$$E\langle\sigma_{N+1}^1\sigma_{N+1}^2(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)\rangle' \simeq E\langle\boldsymbol{m}^1\cdot\boldsymbol{m}^2(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)\rangle \tag{5.52}$$

$$E\langle\boldsymbol{m}^1\cdot\boldsymbol{m}^2(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)\rangle \simeq E\langle\boldsymbol{m}^1\cdot\boldsymbol{m}^2(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)\rangle' \tag{5.53}$$

$$\begin{aligned} &E\langle\boldsymbol{m}^1\cdot\boldsymbol{m}^2(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2)\rangle' \simeq \\ &E\langle(\sum_{k\le M}(\frac{1}{N'}\sum_{i\le N+1}\eta_{i,k}\sigma_i^1)(\frac{1}{N'}\sum_{i\le N+1}\eta_{i,k}\sigma_i^2)) \\ &\qquad(\frac{1}{N'}\sum_{i\le N+1}\sigma_i^1\sigma_i^2)\rangle' := W \end{aligned} \tag{5.54}$$

$$W = E\langle\sigma_{N+1}^1\sigma_{N+1}^2(\sum_{k\le M}(\frac{1}{N'}\sum_{i\le N+1}\eta_{i,k}\sigma_i^1)(\frac{1}{N'}\sum_{i\le N+1}\eta_{i,k}\sigma_i^2))\rangle' \tag{5.55}$$

$$W \simeq E\langle\sigma_{N+1}^1\sigma_{N+1}^2\boldsymbol{m}^1\cdot\boldsymbol{m}^2\rangle' \tag{5.56}$$

$$E\langle\sigma_{N+1}^1\sigma_{N+1}^2\boldsymbol{m}^1\cdot\boldsymbol{m}^2\rangle' \simeq E\langle(\boldsymbol{m}^1\cdot\boldsymbol{m}^2)\rangle^2 \simeq \alpha. \tag{5.57}$$

There $\simeq$ means equality up to lower order terms. The idea of this chain of relations is very simple. The purpose of (5.49) and (5.53) is to replace $\langle\ \rangle$ by $\langle\cdot\rangle'$ for which we have Proposition 5.6. The purpose of (5.50), (5.54) is to create symmetry between the sites, symmetry that is used in (5.51), (5.54). The purpose of (5.56) is to remove the dependence in $\sigma_{N+1}^1, \sigma_{N+2}^2$ in the term next to $\sigma_{N+1}^1\sigma_{N+1}^2$ of (5.55). The key steps are (5.52), (5.57) that rely upon (5.38). The difficulty is that one has to control the error terms with some care. For example, in (5.49), use of (5.38) creates a term

$$E\langle(\frac{1}{N}\sum_{i\leq N}\sigma_i^1\sigma_i^2)^2\boldsymbol{m}^1\cdot\boldsymbol{m}^2\rangle$$

for which we use Holder's inequality to write

$$\begin{aligned}|E\langle(\frac{1}{N}\sum_{i\leq N}\sigma_i^1\sigma_i^2)^2\boldsymbol{m}^1\cdot\boldsymbol{m}^2\rangle| &\leq (E\langle(\boldsymbol{m}^1\cdot\boldsymbol{m}^2)^3\rangle)^{1/3}E\langle(\frac{1}{N}\sum_{i\leq N}\sigma_i^1\sigma_i^2)^3\rangle^{2/3}\\ &\leq L\alpha(E\langle(\frac{1}{N}\sum_{i\leq N}\sigma_i^1\sigma_i^2)^2\rangle)^{2/3}\end{aligned}$$

using Theorem 2.1, and this is

$$\leq L(\alpha^2+\sqrt{\alpha}E\langle(\frac{1}{N}\sum_{i\leq N}\sigma_i^1\sigma_i^2)^2\rangle)$$

which is obviously a lower order term. The other relations are handled in a similar manner. □

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