My favorite problems

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Abstract

I discuss some problems I studied but could not solve.

1 Introduction

I am not a theory builder or a visionary. Much of my mathematical activity was trying to solve specific problems which I learned from others. I studied so many problems that I can almost make statistics. When studying an open problem, most of the time, the effort is largely wasted, as the problem has a negative solution, which is provided by a counter-example to the original question. Sometimes, however, one needs to make some kind of new observation to solve the problem, and if one is lucky this observation develops into a powerful technique or machinery. My experience is that, quite amazingly, the probability that such a miracle occurs is not smaller when the problems look special and specific. It did happen to me several times that such problems were the starting points of very fruitful research directions. For this reason I am not shy to propose several such problems which look more like puzzles, in the sense that the odds seem that once the solution is found (possibly after much struggle) one will not be much wiser. This will be the case for the problems of Section 5. On the other hand, the problems of Section 3 seem more likely to touch central issues, although it could very well happen that even there a simple counter example dismisses the whole story. Fellow mathematicians, such is our life, there is no guaranteed return on investment. As for Section 4, there could also be some potential. And for Section 6, who can tell?

The period following the submission of this paper has been extraordinary. While most of the problems I raised had received no attention whatsoever, several complete solutions or major progresses occurred in a short time. Several will be discussed below. This is not the case of the convolution problem of [T1], which also received a complete solution. The main advance was preformed by Yuansi Chen in [C], and a remaining parasitic logarithmic term was removed in[L-G-F][2].

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[2]It must be said that the result of [L-G-F] is apparently due to an AI agent.

2 Creating convexity in a few steps

Consider a subset A of $\mathbb{R}^N$. Then an easy result of Caratheodory asserts that every point in the convex hull of A is a convex combination of $N+1$ points of A. In flowery words, we can recover the convex hull of A in $N+1$ operations and we cannot do it with fewer operations. But if A is large, can we create a large convex set from A in a bounded number of operations? The canonical notion of "large" is for the natural Gaussian measure γ_N on $\mathbb{R}^N$. In [T2] I raised the following question:

Problem 2.1 *Does there exist an integer n such that whenever A is a closed balanced*[3] *set in $\mathbb{R}^N$ with $\gamma_N(A) \geq 3/4$ then the sum $A_{(n)} = A + \cdots + A$ (n times) contains a convex set C with $\gamma_N(C) \geq 3/4$?*

You notice that I did not dare calling this a conjecture, because I felt that it would be such an extraordinary fact if true. In this statement, the condition $\gamma_N(A) \geq 3/4$ can be replaced by $\gamma_N(A) > \alpha$ for any $\alpha > 1/2$ (changing if necessary the value of n). One simply cannot believe that such a result could be true, until one tries to disprove it. One then realizes how little is understood about the condition $\gamma_N(A) \geq 3/4$. Basically the only examples of non trivial sets A of large measure which I know are based on (extensions of) the following idea: Under the law γ_N the components of a point of $\mathbb{R}^N$ are i.i.d. of law γ_1, so the set A of points $(x_i)_{i\leq N}$ such that the empirical measure $N^{-1}\sum_{i\leq N}\delta_{x_i}$ is close to γ_1 in a suitable sense will be of probability close to 1.[4] This idea is used both in [T2] and [J]. In [T2] it is shown that $n = 2$ does not work. More precisely, given a number $L > 0$, one can find N and a balanced set A with $\gamma_N(A) \geq 3/4$ such that $L(A + A)$ does not contain a large convex set. In [J] it is a more restrictive problem which is considered: Given n the author constructs N and a balanced set $A \subset \mathbb{R}^N$ such that the set $A(n)$ of convex combinations of n points of A does not contain a large convex set. To provide a negative answer to Problem 2.1 one should require that $nA(n)$ does not contain a large convex set. This is a much stronger requirement because blowing up a very small (in the sense of γ_N) convex set by a factor n (or even 2!) may produce a very large one.

It seemed to me that Problem 2.1 is fundamental. It is all the more remarkable that for 30 years after it was printed, there was no sign of interest whatsoever from the rather large group of people focusing on the study of convex sets. However very recently Antoine Song [S] made a breakthrough which led in [H-S-T] to the following amazing result.

[3]That is, $tx \in A$ for $x \in A$ and $|t| \leq 1$.

[4]Such sets are not balanced, but then one replaces them by their balanced hull.

Theorem 2.2 ([H-S-T]) *Whenever the compact set*[5] *A satisfies $\gamma_N(A) \geq 7/8$, the set $5(A + A + A)$ contains a convex set C with $\gamma_N(C) \geq 3/4$.*[6]

We will give part of the proof, which is very indirect. I must stress here that (at least in my style of mathematics) there is quite a difference between being able to prove a theorem (which is still very nice indeed) and really understanding what is going on (which is the source of further progress). This understanding is not yet here. How does C relate to the "structure of A"? One then should ask the following (imprecise) question.

Problem 2.3 *Describe an algorithm which, given a closed set A with $\gamma_n(A) \geq 7/8$, "uses the structure of A" to construct a convex set C as in Theorem 2.2.*

For example, we could dream of finding a function of A and a convex set C such that maximizing it over C would work.

We say that a $\mathbb{R}^N$-valued r.v. G is standard Gaussian if its law is γ_N. The key step to prove Theorem 2.2 is the following amazing and totally unexpected result.

Theorem 2.4 ([H-S-T]) *Consider an $\mathbb{R}^N$ valued r.v. X. Assume that for each convex function f on $\mathbb{R}^N$ we have $\mathsf{E}f(X) \leq \mathsf{E}f(G)$. Then we can find three standard Gaussian r.v.s G_1, G_2, G_3 such that $X = G_1 + G_2 + G_3$.*

Of course G_1, G_2, G_3 are not independent! The proof of Theorem 2.2 uses a wide array of analytical tools, some of which feel like absolute magic.

We will deduce Theorem 2.2 from Theorem 2.4. The core of the proof is the geometric form of the Hahn-Banach Theorem. In a normed space Z, given two disjoint convex sets $\mathcal{C}_0$ and $\mathcal{C}_1$, if one of them is open, we can separate them by a linear functional, that is, there is linear functional φ on Z such that

$$\sup_{f \in \mathcal{C}_0} \varphi(f) \leq \inf_{f \in \mathcal{C}_1} \varphi(f). \tag{2.1}$$

Let us now denote by $\mathcal{C}$ the class of non-negative convex functions f on $\mathbb{R}^N$ such that $f(0) = 0$ and $\mathsf{E}f(G) \leq 1$, so that $\mathcal{C}$ is convex. For each function $f \in \mathcal{C}$ we consider the convex set $C_f = \{f \leq 5\}$, so that $\gamma_N(C_f) \geq 4/5$. Let us denote by B_N a closed ball of $\mathbb{R}^N$ centered at 0 and of radius large enough that $\gamma_N(B_N) \geq 19/20$.

Lemma 2.5 *Consider a symmetric open subset D of $\mathbb{R}^N$ and assume that D does not contain a convex set C with $\gamma_N(C) \geq 3/4$. Consider the compact set $K = B_N \backslash D$. Then for each $f \in \mathcal{C}$ we have $C_f \cap K \neq \emptyset$.*

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[5] It is not required to assume that A is balanced.

[6] It is nice to have explicit small constants. This opens one more silly "best constants" game which we do not enter. We simply mention that a close inspection of the proof allows replacing the set $5(A + A + A)$ by the set $4(A + A + A)$.

Proof: The convex set $B_N \cap C_f$ satisfies $\gamma_N(B_N \cap C_f) \geq 3/4$ so that it is not contained in D. $\square$

Lemma 2.6 *Under the preceding hypotheses, there exists a symmetric r.v. X valued in K such that for each convex function f on $\mathbb{R}^N$ we have $\mathsf{E}f(X/5) \leq \mathsf{E}f(G)$.*

Proof. We will use the geometric form of the Hahn-Banach theorem in the space Z of continuous functions on K, provided with the supremum norm. Consider the convex set $\mathcal{C}_0$ consisting of the restrictions to K of the functions of $\mathcal{C}$. Consider the open convex set $\mathcal{C}_1$ of continuous functions g on K such that $g(y) > 5$ for each $y \in K$. Thus $\mathcal{C}_0$ and $\mathcal{C}_1$ are disjoint by Lemma 2.5, and therefore can be separated by a linear functional φ as in (2.1). If $f \in \mathcal{C}_1$ and $g \geq 0$ it is obvious that $f + \lambda g \in \mathcal{C}_1$ whenever $\lambda \in \mathbb{R}^+$. This implies that $\varphi(g) \geq 0$. In short, there is a positive measure μ such that $\varphi(f) = \int f(y)\mathrm{d}\mu(y)$. We may as well assume that μ is a probability, and (2.1) implies that $\sup_{f \in \mathcal{C}_0} \varphi(f) \leq 5$. Thus a $\mathbb{R}^N$-valued r.v. Y of law μ satisfies $\mathsf{E}f(Y) \leq 5$ whenever $f \in \mathcal{C}$, and by homogeneity $\mathsf{E}f(Y) \leq 5\mathsf{E}f(G)$ whenever f is convex, positive with $f(0) = 0$. By convexity we have $f(Y/5) \leq f(Y)/5$ so that $\mathsf{E}f(Y/5) \leq \mathsf{E}f(G)$. Applying this to the function $\bar{f}$ given by $\bar{f}(x) = f(-x)$ we also have $\mathsf{E}f(-Y/5) \leq \mathsf{E}f(G)$.

Consider now a Bernoulli (= coin-flipping) r.v. η independent of Y so that the r.v. $X = \eta Y$ is symmetric, valued in K and satisfies $\mathsf{E}f(X/5) \leq \mathsf{E}f(G)$ whenever f is convex and non-negative. As there is equality when f is affine, this inequality holds whenever f is convex. $\square$

The following simple fact relies an a compactness argument and will not be detailed.

Lemma 2.7 *Consider a compact set $B \subset \mathbb{R}^N$. If each neighborhood of B contains a convex symetric set C with $\gamma_N(C) \geq 3/4$ so does B.*

Proof of Theorem 2.2. If A is a compact subset of $\mathbb{R}^N$ and V is a neighborhood of $A + A + A$ there exists an open set $A' \supset A$ such that $A' + A' + A' \subset V$. Therefore Lemma 2.7 shows that to prove Theorem 2.2 one may assume that A is open rather than compact. Replacing A by $A \cap (-A)$ we may assume that A is symmetric and $\gamma_N(A) \geq 3/4$.

Assume for contradiction that there exists an open symmetric set A with $\gamma_N(A) \geq 3/4$ such that $5(A + A + A)$ does not contain a convex set C with $\gamma_N(C) \geq 3/4$. Then as we have shown (taking $D = 5(A + A + A)$) there is a symmetric r.v. X valued in $\mathbb{R}^N \setminus 5(A + A + A)$ such that $\mathsf{E}f(X/5) \leq \mathsf{E}f(G)$ for each convex function f. According to Theorem 2.4 we can write $X/5$ as a sum $G_1 + G_2 + G_3$ of three standard Gaussian r.v.s. For each $1 \leq i \leq 3$ we have $\mathsf{P}(G_i \in A) \geq 3/4$ so that

$\mathsf{P}(G_1 + G_2 + G_3 \in A + A + A) \geq 1/4$ and hence $\mathsf{P}(X \in 5(A + A + A)) \geq 1/4$, a contradiction which concludes the proof. $\square$

3 Discrete version

When faced with a (seemingly!) absolutely impregnable problem as Problem 2.1, a good strategy is to invent a simpler problem of the same nature. A central difficulty in Problem 2.1 is that we cannot visualize the space $\mathbb{R}^N$. So, let us replace $\mathbb{R}$ by a set as simple as possible: $\{0, 1\}$. Instead of balanced sets in $\mathbb{R}^N$, identifying $\{0, 1\}^N$ with the set of subsets of $\{1, \ldots, N\}$, we say that the set A is hereditary if $X \in A, Y \subset X \Rightarrow Y \in A$. And we will now define $A_{(n)}$ as the collections of sets of the type $X_1 \cup \cdots \cup X_n$ for $X_1, \ldots, X_n \in A$: the operation of taking union replaces the sum in $\mathbb{R}^N$. To measure the size of A we use the measure $\mu_p = ((1-p)\delta_0 + p\delta_1)^{\otimes N}$ where $0 < p < 1$. (The value of N remains implicit in the notation μ_p. The case of interest is the case p small.) To complete the formulation of the problem, we keep in mind the following very remarkable property of Gaussian measures, see [T4] p. 73.

Theorem 3.1 *There exists a number L with the following property. For any N and any balanced convex set $A \subset \mathbb{R}^N$ with $\gamma_N(A) \geq 3/4$ then the complement of $LA = \{Lx; x \in A\}$ is covered by a countable union of half-spaces H_ℓ with $\sum_{\ell \geq 1} \gamma_N(H_\ell) \leq 1/2$.*

For a subset I of $\{1, \ldots, N\}$ define $H_I = \{X; I \subset X\}$. When working in $\{0, 1\}^N$ rather than in $\mathbb{R}^N$ it does not take much imagination to thinks that the sets H_I are appropriate substitutes for half-spaces. Also, $\mu_p(H_I) = p^{\mathrm{card} I}$. We are then led to the following definition (which is given in [T2] under a different name).

Definition 3.2 ([T2]) *A subset A of $\{0, 1\}^N$ is p-small if $A \subset \bigcup_{\ell \geq 1} H_{I_\ell}$ with*

$$\sum_{\ell \geq 1} \mu_p(H_{I_\ell}) = \sum_{\ell \geq 1} p^{|I_\ell|} \leq 1/2.$$

For a subset A of $\{0, 1\}^N$ let us now denote by $A^{(n)}$ the complement of $A_{(n)}$, that is the collection of subsets of $\{1, \ldots, N\}$ which cannot be covered by n sets in A. We can now state a combinatorical version of Problem 2.1.

Problem 3.3 *Does there exist a number n such that $A^{(n)}$ is p-small whenever $A \subset \{0, 1\}^N$ satisfies $\mu_p(A) \geq 3/4$?*

Of course, it is the same problem whether or not one requires A to be hereditary. This problem is discussed at length in [T3]. It did attract some attention from experts in combinatorics, but, sadly, it remains open. One progress which has been made is that J. Park and H. Pham provided in [P-P] a magnificent proof of an exact analogue to Theorem 3.1.

Theorem 3.4 ([P-P]) *There exists a number L with the following property. Consider a family T of sequences $t = (t_i)_{i \leq N}$ with $t_i \geq 0$. Define the function f_T on $\{0,1\}^N$ by $f_T(X) = \sup_{t \in T} \sum_{i \in X} t_i$, and let $\mathsf{E} f_T = \int f_T(X) \mathrm{d}\mu_p(X)$. Then the set $\{f_T \geq L \mathsf{E} f_T\}$ is p-small.*

A positive solution to Problem 3.3 would imply Theorem 3.4, because the set $A = \{f \leq 4\mathsf{E} f_T\}$ satisfies $\mu_p(A) \geq 3/4$ and $\{f > 4n\mathsf{E} f_T\} \subset A^{(n)}$. However there seems to be no way to go the other direction, as the set A is of a very special type. One could say that Theorem 3.4 assumes that "one already has convexity" but gives no information about "creating convexity".

My feeling is that the main reason that Problem 3.3 remains so mysterious is that we do not know how to approach the operation " taking the union of two sets in A". It seems clear that the resulting set is in a sense "much larger than A" and the challenge is to find a way to express this.[7] In Problem 3.3 we take the union of n sets of A, but I feel that the key would be to understand what happens for $n = 2$. This motivates the following.

Problem 3.5 ([T2]) *Does there exist a number $\alpha > 0$ with the following property: Given any subset A of $\{0,1\}^N$ with $\mu_p(A) \geq 3/4$, then the set $A^{(2)}$ is (αp)-small?*

We leave the reader convince herself that a positive solution of this problem would imply a positive solution of Problem 3.3 (maybe at the cost of replacing 3/4 by a larger number in Problem 3.3.)

It is shown in [T4] how to deduce the following from Theorem 2.2.

Theorem 3.6 ([H-S-T]) *There exists an integer q such that given any subset A of $\{0,1\}^N$ with $\mu_p(A) \geq 1 - 1/q$ then $A^{(q)}$ is p^q-small.*

It is a very striking fact that this combinatorical result is obtained as a consequence of the analytical arguments underlying Theorem 2.4, and it seems a good question to clarify by which mechanism this is possible. Much more is known now by even more recent results. Combining Theorems 3.11 and 3.14 below shows that there exists $\alpha > 0$ such that $A^{(2)}$ is $(\alpha p / \log\log(100/p))$-small.

[7] As we will see later, Chen Li [Li] has recently made a breakthrough on this question.

The previous considerations bring forth a whole line of problems in the same direction. Consider a family $\mathcal{X}$ of subsets of $\{1, \ldots, N\}$. What is the smallest value of p such that whenever $A \subset \{0,1\}^N$ satisfies $\mu_p(A) \geq 3/4$ then there is an element of $A_{(2)}$ which contains an element of $\mathcal{X}$, or equivalently, there is an element of $\mathcal{X}$ contained in the union of two elements of A? For example, assume that $\{1, \ldots, N\}$ is made of q blocks of length k, and that $\mathcal{X}$ consists of the sets which meet each block exactly once. We leave it as a teaser to the reader to prove that any p large enough that $(1-p)^k \leq 1/2$ works. (A stronger result will be proved below, but the present claim is simpler.) Suppose now that $N = M^2$ is a square and think of the N points as a $M \times M$ grid. What happens when $\mathcal{X}$ is the family of sets which meet each line and each column in exactly one point? (Equivalently $\mathcal{X}$ consists of sets of the type $\{(i, \sigma(i)); 1 \leq i \leq M\}$ where σ is a permutation of $\{1, \ldots, M\}$.)

These questions are connected to the idea of "weakly p-small sets" developed in [T3]. To save some space and energy let us define:

Definition 3.7 *A probability measure ν on $\{0,1\}^N$ is α-spread if $\nu(H_I) \leq \alpha^{\mathrm{card} I}$ for each I.*

If $\mathcal{X}$ carries a p-spread probability measure ν and $\mathcal{X} \subset \bigcup_{\ell \geq 1} H_\ell$ then

$$1 \leq \sum_{\ell \geq 1} \nu(H_\ell) \leq \sum_{\ell \geq 1} \mu_p(H_\ell).$$

Thus $\mathcal{X}$ is not p-small.

Definition 3.8 *A weakly p-small set is a set which does not carry a p-spread probability measure.*

Thus a p-small set is weakly p-small. It is explained in [T3] that the distinction between "p-small" and "weakly p-small" parallels the distinction between "cover" and "fractional cover". In [T3] I brazenly conjectured the following:

Conjecture 3.9 *There exists a positive number α such that every weakly p-small set is (αp)-small.*

Not knowing the answer, we may ask the following possibly weaker version of Problem 3.5:

Problem 3.10 *Does there exist a number $\alpha > 0$ with the following property: Given any subset A of $\{0,1\}^N$ with $\mu_p(A) \geq 3/4$, then the set $A^{(2)}$ is weakly (αp)-small.*

It certainly came as a pleasant surprise that Chen Li [Li] recently provided as strong a positive answer as one might have hoped.

Theorem 3.11 ([Li]) *If $\mu_p(A) > (1-p)/(2-p)$ then $A^{(2)}$ is weakly p-small.*

As this result came up days before this paper was sent to the printer, there is no time to assess what are the potential developments. It seems worth to reformulate Theorem 3.11.

Corollary 3.12 *Given any subset A of $\{0,1\}^N$ with $\mu_p(A) > (1-p)/(2-p)$ and any family $\mathcal{X} \subset \{0,1\}^N$ which carries a p-spread probability measure we can find $X \in \mathcal{X}$ which can be covered by two elements of A.*

At first sight, the definition of α-spread probability measures looks like an appealing concrete starting point for our investigations, but in fact such probability measures seem very hard to understand in general. The following result of Chen Li [Li] might sometimes provide a line of attack.

Theorem 3.13 ([Li]) *If ν is a p-spread probability measure, there exists three $\{0,1\}^N$-valued r.v.s V, W_1, W_2 respectively of laws ν, μ_p, μ_p such that $V \subset W_1 \cup W_2$.*

In view of Theorem 3.11 we would have a positive answer to our main question, Problem 3.5, if we had a positive answer to Conjecture 4.3. The current best known result is as follows.

Theorem 3.14 ([P]) *There exists a number $\alpha > 0$ such that any p-weakly small set is $(\alpha p/\log\log(100/p))$ – small.*

The gap between this result and Conjecture 4.3 is indeed small![8]

4 Projections

This section is a continuation of the previous one, but it develops a new direction.

As there is no obvious way to handle the set $A^{(2)}$ one may remember the elementary fact that if $\mu_{1/2}(A) \geq 1/2$ then $\{1, \dots, N\} \in A^{(2)}$. This suggests a more general approach to Problem 3.5. For $X \in \{0,1\}^N$ let us denote by P_X the projection from $\{0,1\}^N$ to $\{0,1\}^X$ where now X is thought of as a subset of $\{1, \dots, N\}$. Let us denote by θ_X the uniform measure on $\{0,1\}^X$.

Problem 4.1 *Does there exist a number $\alpha > 0$ such that whenever $A \subset \{0,1\}^N$ is hereditary and satisfies $\mu_p(A) \geq 3/4$ then the set $\{X \in \{0,1\}^N ; \theta_X(P_X(A)) < 1/2\}$ is (αp)-small?*

[8]But, as the saying goes, removing the last log is often exponentially hard. So think about the last loglog!

A positive solution would imply a positive solution to Problem 3.5. To provide a position answer to Problem 4.1 one has to show that if a class $\mathcal{X} \subset \{0,1\}^N$ is not (αp)-small, then it contains a set X such that $\theta_X(P_X(A)) \geq 1/2$. The best known result in that direction seems to be that (if α is small enough) one can find $X \in \mathcal{X}$ and $Y \in A$ such that $\mathrm{card}(Y \cap X) \geq \mathrm{card}\, Y/2$.

An obvious line of approach is to replace in Problem 4.1 the requirement "(αp)-small" by "weakly (αp)-small". As the map $X \mapsto \theta_X(P_X(A))$ is decreasing, in view of Theorem 3.13, one may also ask the following.

Problem 4.2 *Does there exist a number $\alpha > 0$ such that whenever the $\{0,1\}^N$-valued r.v.s W_1, W_1 have law $\mu_{\alpha p}$ and $X = W_1 \cup W_2$, the r.v. $\theta_X(P_X(A))$ takes a value $> 1/2$?*

It never hurts to study examples. For the two classes $\mathcal{X}$ considered at the end of the previous section, the uniform probability on $\mathcal{X}$ is $1/k$-spread in the first case and in the second case it is C/M-spread for some number C. In the next two results, ν is the uniform measure on $\mathcal{X}$.

Proposition 4.3 *Consider two integers k, q, set $N = kq$ and think of $\{1, \ldots, N\}$ as made of q blocks of size k. Consider the class $\mathcal{X}$ of sets that meet every block in exactly one point. Then as soon as $(1-p)^k \leq 1/2$, for each hereditary set A we have*

$$\int \theta_X(P_X(A))\mathrm{d}\nu(X) \geq \mu_p(A) \; . \tag{4.1}$$

In particular we can find $X \in \mathcal{X}$ such that $\theta_X(P_X(A)) \geq \mu_p(A)$.

The way to prove a statement such as (4.1) is to prove that the measure μ_p can be "pushed down" to the measure η given by $\eta(A) = \int \theta_X(P_X(A))\mathrm{d}\nu(X)$, that is to prove that there is a disintegration $\eta = \int \eta_Y \mathrm{d}\mu_p(Y)$ where the probability measure η_Y on $\{0,1\}^N$ is supported by the subsets of Y. It is then clear that the proof reduces to the case $q = 1$ where it is really easy. The proof of the following is far less obvious and is left as a challenge to the reader.

Proposition 4.4 *There exists a number L with the following property. Assume that $N = M^2$ and think of $\{1, \ldots, N\}$ as $\{1, \ldots, M\}^2$. Consider the class $\mathcal{X}$ of subsets of $\{1, \ldots, M\}^2$ which meet each line and each column is exactly one point. Then as soon as $p \geq L/M$, if $A \subset \{0,1\}^N$ is hereditary then (4.1) holds.*

Of course, based on these examples, one may become absurdly greedy and ask the following.

Problem 4.5 *Does there exist a number $\alpha > 0$ such that whenever ν is an (αp)-spread probability measure (4.1) holds for each hereditary set A?*

Of course, as in Problem 4.2 one may ask what happens when ν is the law of $X = W_1 \cup W_2$ where the $\{0,1\}^N$-valued r.v.s W_1, W_1 have law $\mu_{\alpha p}$.

While I raised most of the previous questions a long time ago, while writing this paper I realized that even when $p = 1/2$, I know very little about the size of the projections of a large set $A \subset \{0,1\}^N$. This led me to raise the following problems. Frédéric Ouimet [O] submitted them to ChatGPT Pro, which provided a complete positive solution to the first two problems, far stronger than what the problems ask [C2], [C3]. I felt I had a lot to learn from these solutions, and I believe that it is a worthy project to write them in a polished human-friendly way, to understand them fully and to pursue the direction if possible.

Problem 4.6 *For $0 < \varepsilon < 1/2$, does there exist a number $\alpha(\varepsilon) > 0$ with the following property. Whenever $A \subset \{0,1\}^N$ satisfies $\mu_{1/2}(A) \ge 3/4$, then the set $\{X \in \{0,1\}^N ; \theta_X(P_X(A)) < 1 - \varepsilon\}$ is $\alpha(\varepsilon)$-small?*

Note that here A is not assumed to be hereditary. However it is enough to consider the case where A is hereditary.

Lemma 4.7 *We do not weaken Problem 4.6 by assuming A hereditary.*

This is proved using the time-honored "push-down method". Given $1 \le i \le N$ and $A \subset \{0,1\}^N$ we define A^i as follows: whenever $i \in X \in A$ and $X \setminus \{i\} \notin A$ we remove X from A and we add $X \setminus \{i\}$ to A. Thus $\mathrm{card} A^i = \mathrm{card} A$. On the other hand, it is not difficult to show that $\nu_X(P_X(A^i)) = \nu_X(P_X(A))$.[9]Iterating the push-down method for each i proves the lemma.

In order to understand what is going on in Problem 4.6 a first goal should be to find examples proving upper bounds for $\alpha(\varepsilon)$. Noga Alon showed me the following, which proves only the very weak bound $\alpha(\varepsilon) \le L/(\log\log(1/\varepsilon))$ for small ε. Consider an integer k not too small and $N = k2^k$. Think of $\{1, \dots, N\}$ as the union of 2^k blocks of length k. Consider the set A consisting of subsets of $\{1, \dots, N\}$ which miss at least one these blocks, so that for large k we have $\mu_{1/2}(A) = 1 - (1 - 2^{-k})^{2^k} \ge 1/2$. Consider then the class $\mathcal{X}$ of subsets of $\{1, \dots, N\}$ which meet each block in exactly one point. Then for each $X \in \mathcal{X}$ we have $\nu_X(P_X(A)) = 1 - 2^{-2^k}$.The set X is not $1/k$-small because the uniform probability on X is $1/k$-spread.

Certainly one should ask what happens when in Problem 4.6 one replaces $\{0,1\}$ by a more general probability space, say $[0,1]$ with Lebesgue's measure. For $X \subset \{1, \dots, N\}$ we denote P_X the projection from $[0,1]^N$ to $[0,1]^X$. We denote by λ_N and λ_X the natural measures on $[0,1]^N$ and $[0,1]^X$.

[9]In fact, for $i \in X$ the operation "pushdown at i" commutes with the projection P_X.

Problem 4.8 *Does there exist a number $\alpha > 0$ such that whenever $A \subset [0,1]^N$ satisfies $\lambda_N(A) \geq 3/4$ then the set of X for which $\lambda_X(P_X(A)) < 9/10$ is α-small?*

To understand better this problem, it helps to consider the case where $A = [0, 1 - 1/(5N)]^N$. This example shows that if in Problem 4.8 we rather consider the set of X for which $\lambda_X(P_X(A)) < 1 - \theta$ we cannot expect better that this set being α-small for α of order θ.

These problems are quite far from Problem 2.1, and it seems a worthy project to investigate what happens if we come back to $\mathbb{R}^N$, using now projections on a subspace of lower dimension. For a subspace W of $\mathbb{R}^N$ we denote by P_W the orthogonal projection of $\mathbb{R}^N$ on W and by γ_W the canonical Gaussian measure on W.

Problem 4.9 *Is it true the for some constant L, if $A \subset \mathbb{R}^N$ satisfies $\gamma_N(A) \geq 3/4$, then for $M \leq N - L\sqrt{N}$ the set of spaces W of $\mathbb{R}^N$ of dimension M for which $\gamma_W(P_W(A)) \leq 9/10$ is extremely small?*

That the condition $M \leq N - L\sqrt{N}$ is necessary is shown by the case where A is a ball centered at the origin. Part of the problem is to invent the appropriate concept of "extremely small" which is relevant here, and which should replace the concept of p-small sets previously used. Chat GPT performed the first step and showed that the proportion of such spaces is very small [C4]. Possibly a much stronger result holds. A natural question is whether one can do even better when one assumes A to be convex balanced.

5 Matchings

The basic object of this section is a sequence $(X_i)_{i \leq N}$ of independent r.v.s uniformly distributed in the unit square $[0,1]^2$. There are typically regions of the unit square which have a deficit or an excess of points and our goal is to quantify this in different ways. This material is extensively discussed in the monograph [T4] to which we refer for all references. Our goal is simply to provide an overview of the main remaining problems. These problems touch some of the central issues of the theory of stochastic processes, so it is not possible to have an in depth discussion of the underlying issues. Rather, we simply try to hook a reader in considering these fascinating questions.

The basic idea to measure the irregularities of the set $\{X_i; i \leq N\}$ is to consider another independent sequence $(Y_i)_{i \leq N}$ and to try to match the points X_i with the points Y_j in a way that two points that get matched are close to each other. Such a matching is given by a permutation π of the set $\{1, \ldots, N\}$.[10] The goal is to

[10] We will lighten the expositions by calling such a permutation "a matching"

prove the existence of a matching in a manner that the distances $d(X_i, Y_{\pi(i)})$ are "small". Two different objectives are to minimize the sum of these distances, or their maximum.

Theorem 5.1 (The Ajtai-Komlós-Tusnády matching theorem) *With high probability we have* $\inf_\pi \sum_{i \le N} d(X_i, Y_{\pi(i)}) \le L\sqrt{N \log N}$.

Here and below L is a universal constant (i.e. a number independent of N) and "with high probability" means "with probability going to 1 as $N \to \infty$" (much more precise statements are available). So, the average distance between X_i and $Y_{\pi(i)}$ is $\le L\sqrt{\log N}/\sqrt{N}$. The factor $1/\sqrt{N}$ is the natural scaling factor, and the factor $\sqrt{\log N}$ reflects the irregularities in the distribution.

Theorem 5.2 (The Leighton-Shor matching theorem) *With high probability we have* $\inf_\pi \max_{i \le N} d(X_i, Y_{\pi(i)}) \le L(\log N)^{3/4}/\sqrt{N}$.

These unexpected powers of $\log N$ are optimal. A striking feature here is that Theorems 5.1 and 5.2 are obtained by rather different proofs, suggesting the obvious question as to whether we can simultaneously control the sum of the distances $d(X_i, Y_{\pi(i)})$ and their maximum, which I raised a long time ago.

Problem 5.3 *Is it true that with high probability we can find a matching π such that* $\sum_{i \le N} d(X_i, Y_{\pi(i)}) \le L\sqrt{N \log N}$ *and* $\max_{i \le N} d(X_i, Y_{\pi(i)}) \le L(\log N)^{3/4}/\sqrt{N}$*?*

The appeal of this question is that it qualified as "the simplest open problem about matchings". It is very impressive that ChatGPT Pro could produce a highly elaborate solution to this problem [C5]. This solution does the job, but does not seem to make progress on the real issues, the considerably more ambitious questions (whose solution is possibly far more difficult) we will consider later.[11]

Let us now denote by X_i^1 and X_i^2 the coordinates of X_i. Peter Shor has discovered the following striking improvement of Theorem 5.1.

Theorem 5.4 (Shor's matching theorem) *With high probability we can find a matching π such that* $\sum_{i \le N} |X_i^1 - Y_{\pi(i)}^1| \le L\sqrt{N \log N}$ *and* $\max_{i \le N} |X_i^2 - Y_{\pi(i)}^2| \le L\sqrt{\log N}/\sqrt{N}$.

This improves Theorem 5.1 which only asserts that $\sum_{i \le N} |X_i^2 - Y_{\pi(i)}^2| \le L\sqrt{N \log N}$. We have replaced the control of the sum of these quantities by the control of each of them. Concerning the quantities $|X_i^1 - Y_{\pi(i)}^1|$ it is possible to do far better than controlling just their sum.

[11] When, and if AI is able to solve such problems, very serious questions will arise about the role of human mathematicians.

Conjecture 5.5 *With high probability there exists a matching π such that*

$$\sum_{i\le N}\exp\Big(\frac{\sqrt{N}}{L\sqrt{\log N}}|X_i^1-Y_{\pi(i)}^1|\Big)^2\le 2N \tag{5.1}$$

and $\max_{i\le N}|X_i^2-Y_{\pi(i)}^2|\le L\sqrt{\log N}/\sqrt{N}$.

One can use the inequality $\exp x\ge 1+x$ to compare this with Theorem 5.4. The best I can do in the direction of Conjecture 5.5 is to prove a result where in (5.1) the exponent 2 is replaced by $\alpha<1/2$ (and where L is replaced by a number depending on α).

Conjecture 5.5 is a special case of the Ultimate Matching conjecture stated below, but there is a very specific reason why we mention it separately: there is a clear road to attack it, and the road block which prevents its solution looks more technical than conceptual.

There is a deep link between matching problems and discrepancy problems. We explain first what these are. Consider a class $\mathcal{F}$ of functions on a probability space (Ω,P), all of mean 0, and an i.i.d. sequence $(X_i)_{i\le N}$ in Ω distributed according to P. A discrepancy bound is a bound on $\sup_{f\in\mathcal{F}}|\sum_{i\le N}f(X_i)|$. Such bounds quantify "how well the empirical mean approaches the true mean" uniformly on the class $\mathcal{F}$, and their study is a central topic of probability. The standard method to obtain discrepancy bounds is to use tail inequalities (in particular Bernstein's inequality) and chaining. The possibility of using chaining requires a certain control of the size of $\mathcal{F}$ seen as a subset of $L^2(\mathsf{P})$. It is a very beautiful fact that both Theorems 5.1 and 5.2 are obtained through a discrepancy result (and duality arguments) and that in both cases the required smallness of the corresponding class $\mathcal{F}$ is deduced from the same abstract result on ellipsoids in Hilbert space (the "Ellipsoid theorem" of [T4]).

The appeal of Conjecture 5.5 is that it boils down to a discrepancy problem, as is explained in [T4] (showing this requires non trivial work). We state this problem now. Let us consider an integer $p\ge 1$ and the grid $G=\{1,\dots,2^p\}^2$. The components of a point u in G are denoted by k,ℓ. We denote by P the uniform probability on G. Consider the function θ on $\mathbb{R}^+$ given by $\theta(x)=x\log(x+3)$. Consider the class $\mathcal{F}$ of functions f on G for which $\sum_{u\in G}f(u)=0$ and which satisfy the condition

$$\sum\theta(|f(k+1,\ell)-f(k,\ell)|)+\sum|f(k,\ell+1)-f(k,\ell)|\le 2^{2p}\,. \tag{5.2}$$

Here, the first sum is over $1\le k\le 2^p-1$ and $1\le\ell\le 2^p$, whereas the second sum is over $1\le k\le 2^p$ and $1\le\ell\le 2^p-1$.

It is claimed in [T4] that to prove Conjecture 5.5 it suffices to prove[12] (an appropriate version of) the following, where the random variables U_i are independent uniform on G.

Conjecture 5.6 *For $N \geq 2^{2p}$ with probability $\geq 1 - 2^{-p}$ we have*

$$\sup_{f\in\mathcal{F}} \Big| \sum_{i\leq N} f(U_i) \Big| \leq L\sqrt{Np}2^p \ .$$

The difficulty in approaching this result is that I do not even understand what is the true size of $\mathcal{F}$ seen as a subset of $L^2(\mathsf{P})$. We can formulate the question as follows, where $(g_u)_{u\in G}$ are independent standard Gaussian r.v.s.

Conjecture 5.7 *It holds that*

$$\mathsf{E}\sup_{f\in\mathcal{F}} \Big| \sum_{u\in G} g_u f(u) \Big| \leq L\sqrt{p}2^{2p} \ . \tag{5.3}$$

This is a question about the supremum of a concrete Gaussian processes. The theory of Gaussian processes has reached a very satisfactory state, in the sense that very precise results are known in complete generality, see [T4]. Unfortunately, as the previous question shows there is no magic wand to understand the combinatorics in concrete examples. Related to Conjecture 5.7 is the question of evaluating the left-hand side of (5.3) when the condition (5.2) is replaced by

$$\sum \theta_1(|f(k+1,\ell) - f(k,\ell)|) + \sum \theta_2(|f(k,\ell+1) - f(k,\ell)|) \leq 2^{2p}$$

for two functions θ_1, θ_2. To show how matters are subtle, when $\theta_1(x) = \theta_2(x) = x$, in the right-hand side of (5.3) it is not possible to do better then $Lp^{3/4}2^{2p}$. This can be shown using the same construction which proves that the power 3/4 is necessary in Theorem 5.2.

We are finally ready to state the following lovely (and beloved) main conjecture of this section.

Conjecture 5.8 (The Ultimate Matching Conjecture) *Consider $\alpha_1,\ \alpha_2 > 0$ with $1/\alpha_1 + 1/\alpha_2 = 1/2$. Then with high probability there exists a matching π such that for $j = 1, 2$ we have*

$$\sum_{i\leq N} \exp\Big[\Big(\frac{\sqrt{N}}{L\sqrt{\log N}}|X_i^j - Y_{\pi(i)}^j|\Big)^{\alpha_j}\Big] \leq 2N.$$

[12]It might require some work to figure out all the details, but the claim is plausible!

Conjecture 5.5 is the special case $\alpha_1 = 2, \alpha_2 = \infty$. The case $\alpha_1 = \alpha_2 = 4$ would provide a very neat generalization of Theorems 5.1 and 5.2.

When working on an open problem, there is always the chance that one is lead to discover a powerful new method, but still the odds seems that Conjecture 5.8 is just a very tough puzzle to crack.

6 From a set to its convex hull

Since so much progress was made while this paper was being prepared, I will return to a forty years old question on which no progress has been made.[13] Recall that we denote by (x, y) the dot product on $\mathbb{R}^N$. For $x \in \mathbb{R}^N$ the function $X_x : y \mapsto (x, y)$ on $\mathbb{R}^N$ can be seen as a r.v. on the probability space $(\mathbb{R}^N, \gamma_N)$. Given a (finite) subset T of $\mathbb{R}^N$ we can then consider the process $(X_t)_{t\in T}$. The importance of this canonical construction stems from the fact that all Gaussian processes arise this way. The set T is provided with a canonical distance d induced by the Euclidean distance in $\mathbb{R}^N$. As explained in considerable detail in [T4], a fundamental fact about Gaussian processes (which is at the root of Theorem 7) is that the probabilistic quantity $\mathsf{E}\sup_{t\in T} X_t$ can essentially be computed as a function of the geometry of the metric space (T, d). There exists a certain quantity $\gamma_2(T, d)$, defined only in terms of the metric d such that for some universal constant L we have

$$\frac{1}{L}\gamma_2(T, d) \le \mathsf{E}\sup_{t\in T} X_t \le L\gamma_2(T, d). \tag{6.1}$$

The fun starts when one observes that $\sup_{t\in T} X_t = \sup_{t\in \mathrm{conv}T} X_t$. Consequently (6.1) implies that for a universal constant L'' one has

$$\gamma_2(\mathrm{conv}T, d) \le L''\gamma_2(T, d). \tag{6.2}$$

This is a purely geometrical statement.

Problem 6.1 *Find a purely geometrical proof.*

There is an intrinsic difficulty in Problem 6.1: The geometry of the space $(\mathrm{conv}T, d)$ does not depend only on the geometry of the space (T, d). To see this, consider a set T_n which is a subset of the unit ball of $\mathbb{R}^n$ maximal with respect to the property that any two points of T_n are at respective distances $\ge 1/4$. We leave as a teaser to the reader the fact that $\mathrm{card}T_n \le 5^n$ and that the ball centered at the origin

[13] Considering this question was immensely beneficial. It is while trying again many years after proving the majorizing measure theorem that I discovered the idea of generic chaining and the quantity $\gamma_2(T, d)$.

of radius 1/2 is contained in the convex hull of T_n, and thus that $\mathsf{E}\sup_{t\in T_n} X_t$ is of order $\sqrt{n}$. Letting $M = \mathrm{card} T_n$, consider the set T_M consisting of the first M vectors of the canonical basis of $\mathbb{R}^M$. Then the metric spaces (T_n, d) and (T_M, d) are basically the same (they are Lipschitz isomorphic with constants independent of n) but the reasons why $\gamma_2(\mathrm{conv} T_n, d)$ and $\gamma_2(\mathrm{conv} T_M, d)$ are of order $\sqrt{n}$ are very different (as the reader will learn while studying the relevant sections of [T4]).

A really satisfactory answer would be to prove 6.2 for subsets of a class of normed space larger than the Euclidean spaces.[14] In [T4] Section 2.11, it was suggested to investigate the case of 2-smooth Banach spaces, but this is ruled out by the recent construction of Zhang and Zhao [Z-Z] who proved that (6.2) fails when the space is $\ell^p(\mathbb{N})$, $p > 2$. Following [Z-Z], for a normed space W let us define

$$C(W) = \sup\Big\{\frac{\gamma_2(\mathrm{convT}, d)}{\gamma_2(T, d)}; T \text{ finite}\Big\}.$$

A natural question is then:

Problem 6.2 *Is it true that W is Lispchitz isomorphic to a subspace of a Hilbert space whenever $C(W) < \infty$?*

Wouldn't a positive answer be a fun geometric characterization of Hilbert spaces?[15] A positive answer would also indicate that there may not be a clean answer to Problem 6.1.

A related question is as follows. For $\ell \le M$ consider a (finite) subset T_ℓ of $\mathbb{R}^N$. Consider the set $T = \prod_{\ell\le M} T_\ell \subset \mathbb{R}^{NM}$. It should then be rather obvious from (6.1) that

$$\gamma_2(T, d) \le L \sum_{\ell\le M} \gamma_2(T_\ell, d),$$

but I would like to see a "geometric" proof of this. More generally, given metric spaces (X_ℓ, d_ℓ) for $\ell \le M$, one may consider the distance d on $X = \prod_{\ell\le M} X_\ell$ given by (with obvious notation)

$$d(x, y)^2 = \sum_{\ell\le M} d_\ell(x_\ell, y_\ell)^2,$$

and, given subsets $T_\ell \subset X_\ell$ investigate various measures of "size" of $T = \prod_{\ell\le M} T_\ell$ given information on the "sizes" of the sets T_ℓ.[16]

[14] This would ensure that Gaussian processes are not used at all for the proof.

[15] A finite-dimensional version of Problem 6.2 would be to ask if given a number $A > 0$ there exists a number B such that W is B-isomorphic to Euclidean space whenever W is finite-dimensional and $C(W) \le A$.

[16] Although this is a natural question, the chances as to whether this is a promising direction are hard to assess.

7 Addendum: alternate proof of Theorem 2.2

A previous version of Problem 2.3 asked for an algorithmic proof of Theorem 2.2. Frédéric Ouimet prompted Chat GPT Astra, which produced such a proof [C1]. Trying to decipher that proof I realized that one can write the argument of Theorem 2.2 in an (apparently?) more constructive way. At the root is again the Hahn-Banach theorem, but now in the form of the minimax theorem: If H and $\mathcal{C}$ are convex compact, and ψ is a continuous bilinear functional on $H \times \mathcal{C}$ then

$$\sup_{f \in H} \inf_{\mu \in \mathcal{C}} \varphi(f, \mu) = \inf_{\mu \in \mathcal{C}} \sup_{f \in H} \varphi(f, \mu).$$

The following is elementary.

Lemma 7.1 *The set H of convex symmetric functions $f \geq 0$ on $\mathbb{R}^N$ such that $\mathsf{E} f(G) \leq 1$ is compact for the topology of uniform convergence on compact sets.*

Consider a symmetric open set A with $\gamma_N(A) \geq 3/4$ and assume that $K := B_N \setminus 5(A + A + A) \neq \emptyset$. Denote by $\mathcal{C}$ the convex compact set of probability measures on K which are invariant under the symmetry $x \mapsto -x$, and consider the bilinear functional φ on $H \times \mathcal{C}$ given by

$$\varphi(f, \mu) = \int f(x/5) \mathrm{d}\mu(x).$$

Lemma 7.2 *Consider $\mu \in \mathcal{C}$. (a) There is a convex function f such that $0 < \mathsf{E} f(G) \leq \int f(x/5) \mathrm{d}\mu(x)$. (b) In fact, we can find $f \in H$ with $\varphi(f, \mu) \geq 1$.*

Proof (a) Consider a r.v. X of law μ. If (a) fails, for each convex function f we have $\mathsf{E} f(X/5) \leq \mathsf{E} f(G)$, so that by Theorem 2.4 we can find three standard Gaussian r.v.s G_1, G_2, G_3 such that $X/5 = G_1 + G_2 + G_3$, and then $P(X/5 \in A + A + A) \geq 1/4$, which is absurd.
(b) Consider the function f provided by (a). Without loss of generality we may assume $f(0) = 0$. Then the function f^* given by $2f^*(x) = f(x) + f(-x)$ also satisfies $0 < \mathsf{E} f^*(G) \leq \int f^*(x/5) \mathrm{d}\mu(x)$, and is symmetric, so that it is ≥ 0. The function λf^* where λ is such that $\lambda \mathsf{E} f^*(G) = 1$ has the desired properties. $\square$

Proof of Theorem 2.2. The minimax theorem provides us with a function $\bar{f} \in H$ such that $\varphi(\bar{f}, \mu) \geq 1$ for each $\mu \in \mathcal{C}$. Consider $x \in K$ and $\nu = (\delta_x + \delta_{-x})/2$, so that since $\bar{f}$ is symmetric we have $\varphi(\bar{f}, \nu) = \bar{f}(x/5) \geq 1$ and thus $\bar{f}(x) \geq 5\bar{f}(x/5) \geq 5$. The convex set $C = \{\bar{f} < 5\} \cap B_N$ does not meet K so it is contained in $5(A + A + A)$. Moreover $\gamma_N(\{\bar{f} \geq 5\}) \leq 1/5$ so that $\gamma_N(C) \geq 3/4$. $\square$

This approach raises a potentially interesting question. Given a probability measure μ on $\mathbb{R}^N$, how to we relate the quantity $\sup\{\int f \mathrm{d}\mu; f \in H\}$ with "the structure of μ"?

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