F. Some Comments on the Discrepency Problems of Chapter 17

The purpose of this appendix it to bring forward some features of one of the major open problems in this book, namely Problem 17.2.2. The thoughts I will present occurred to me too late to be included in the printed version of the book. I would also like to make the structure of this problem clearer, since, as it is stated now it is necessary to have studied in detail Section 17.2. to have a chance to make sense of it.

- The link between matching theorems and discrepency theorems is explained in detail in Chapter 4.
- When studying matchings of a random i.i.d. sample of N points in $[0,1]^2$ (with evenly distributed points), we are typically not interested in what happens at a scale less than $1/\sqrt{N}$[1] so it is natural to replace the underlying probability space $[0,1]^N$ by a discrete version $G = \{1, \ldots, 2^p\}^2$, say with the uniform probability μ.
- The heart of the problem is then to study the size of certain classes of functions $\mathcal{H}$ on G (as subsets of $L^2(G, \mathrm{d}\mu)$)

Consider $a > 0$ and the function $\varphi_a(x) = |x|(\log(2 + |x|))^a$. Consider the class $\mathcal{H}_{a,b}$ of functions h on G which satisfy the following conditions. First, h is "0 on the boundary of G":

$$(k,\ell) \in G\ ,\ k \in \{1, 2^p\} \Rightarrow h(k,l) = 0$$

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Next

$$\sum_{k,\ell} \varphi_a(h(k+1,\ell) - h(k,\ell)) \le 2^{2p}\ . \tag{F.1}$$

Here of course the summation is over $1 \le k \le 2^p - 1$ and $1 \le \ell \le 2^p$. Also

$$\sum_{k,\ell} \varphi_b(h(k,\ell+1) - h(k,\ell)) \le 2^{2p}\ , \tag{F.2}$$

where the summation is over $1 \le k \le 2^p$ and $1 \le \ell \le 2^p - 1$.

Research problem F.0.1. What is the order of $\gamma_2(\mathcal{H}(a,b))$??

[1] or even $\sqrt{\log N}/\sqrt{N}$, etc. but the problem we will raise is not sensitive to that.

let me now explain the point of this problem. Consider the much smaller class $\mathcal{H}_{\rm Lip}$ of functions that are 0 on the boundary of G and satisfy $|h(k+1,\ell)-h_(k,\ell)|\le 1$ and $|h(k,\ell+1)-h(k,\ell)|\le 1$. Then the essence of Section 4.6 is to prove that $\gamma_2(\mathcal{H}_{\rm Lip})\ge \sqrt{p}2^{2p}/L$. Consequently

$$\gamma_2(\mathcal{H}_{a,b})\ge \frac{1}{L}\sqrt{p}2^{2p}\ .$$

I believe[2] that if one could show that

$$\gamma_2(\mathcal{H}_{0,1/2})\le L\sqrt{p}2^{2p}\ . \tag{F.3}$$

one could prove Conjecture 17.1.4. I have never written down the argument in detail (so please do not embarrass me by asking) but the path should be as follows. First prove a suitable version of Theorem 17.2.1, where the class $\mathcal{H}$ there is replaced by the class $\mathcal{H}_{0,1/2}$, using that $\gamma_2(\mathcal{H}_{0,1/2})\le L\sqrt{p}2^{2p}$ and Theorem 4.5.13. Then use the method sexplained in [169] Section 3.5.

It is a highly non-trivial fact that $\gamma_2(\mathcal{H}_{0,b})$ is of a larger order than $\sqrt{p}2^{2p}$ for $b<1/2$. The reason for this is that Theorem 4.5.13 fails for $\alpha>2$ for otherwise with high probability we would get a matching π for which $\sup_{i\le N}|X_i^1-Y^1_{\pi(i)}|\le K(\log N)^{1/2+1/\alpha}N^{-1/2}$ and $\sup_{i\le N}|X_i^2-Y^2_{\pi(i)}|\le K\sqrt{\log N}N^{-1/2}$. Such a matching does not exist, basically for the same reason that the Leighton-Shor matching theorem is sharp (one cannot do better than the exponent 3/4 there) and a tree witnessing that $\gamma_2(\mathcal{H}_{0,b})$ is of larger order than $\sqrt{p}2^{2p}$ can be constructed using the ideas of the lower bound of the Leighton-Shor matching theorem.

I write this appendix because I sort of outsmarted myself. I designed a "dimension independent way" to get the factors $\sqrt{\log N}$ occurring in matching theorems by introducing the functionals $\gamma_{\alpha,\beta}$ of (4.5). It then follows from Lemma 4.1.3 that for a class $\mathcal{H}$ of functions on G we have

$$\gamma_2(\mathcal{H})\le L\sqrt{p}\gamma_{2,2}(\mathcal{H})\ . \tag{F.4}$$

This is basically obtained by applying the Cauchy-Schwarz inequality, and this result is sharp when in the application of the Cauchy-Schwarz inequality we are basically in the case of equality. A typical application of this result is to the case $\mathcal{H}=\mathcal{H}_{\rm Lip}$ because $\gamma_{2,2}(\mathcal{H}_{\rm Lip})\le L2^{2p}$. (This last claim being a discrete version of Lemma 4.5.9.)

I plan to explain the following (a weaker fact is claimed in [153]):

Proposition F.0.2. *If* $\gamma_{2,2}(\mathcal{H}_{a,b})\le L2^{2p}$ *then* $a+b\ge 1$.

This does *not* contradict (F.3) because we are not at all in the situation where the use of the Cauchy-Schwarz inequality in (F.4) would be sharp and this inequality cannot be reversed.

[2] A similar claim is made right after (17.18).

Before proving Proposition F.0.2, I will first prove a "continuous version" of it. Let us fix $a, b \geq 0$ and consider the class $\bar{\mathcal{H}}_{a,b}$ of functions on $[0,1]^2$ which satisfy $\int \varphi_a(h)\mathrm{d}\mu \leq 1$ and $\int \varphi_b(h)\mathrm{d}\mu \leq 1$, and are zero on the boundary of $[0,1]^2$.

Proposition F.0.3. *If* $\gamma_{2,2}(\bar{\mathcal{H}}_{a,b}) < \infty$ *then* $a + b \geq 1$.

Proof. First, for any set $\mathcal{H}$ is a Hilbert space we have $\sup_n 2^{n/2} e_n(\mathcal{H}) \leq L\gamma_{2,2}(\mathcal{H})$. This is a variation on (2.57). Assuming that $a + b < 1$ we will prove that then

$$\sup_n 2^{n/2} e_n(\bar{\mathcal{H}}_{a,b}) = \infty \;, \tag{F.5}$$

and this will prove that $\gamma_{2,2}(\bar{\mathcal{H}}_{a,b}) = \infty$.

Given a number $0 < u < 1$ consider the function f_u on $[0,1]$ such that $f_u(u) = 1$, $f_u(0) = 0$, $f_u(x) = 0$ for $x \geq 2u$ and f_u piece wise afine. Consider the function $g_{u,v} = f_u \otimes f_v$ for $0 < u, v \leq 1$ so that $\|g_{u,v}\|_2$ is about $\sqrt{uv}$, where the norm is in $L^2([0,1]^2)$. Considering $c > 0$ the quantity $\int \varphi_a(\partial_x c g_{u,v})\mathrm{d}\mu$ is about $uv \times c/u \log(2 + c/u)^a$=$cv \log(2 + c/u)^a$ and $\int \varphi_b(\partial_y c g_{u,v})\mathrm{d}\mu$ about $cu \log(2 + c/v)^b$. For c large, both these quantities are about 1 for $v = 1/(c \log(2 + c)^a)$ and $u = 1/(c \log(2 + c)^b)$. Thus the function $c g_{u,v}$ is of L^2 norm about $c\sqrt{uv}$= $1/(\log(2 + 1/c))^{(a+b)/2})$. Now the function $c g_{u,v}$ is supported by the rectangle $[0, 2u] \times [0, 2v]$, and we can find about $1/(uv) \geq c^2$ disjoint translates of this rectangle in the unit square. That means that in we can find at least c^2 points in $\bar{\mathcal{H}}_{a,b}$ which are at mutual distance $\geq 1//(\log(2 + c))^{(a+b)/2})$. Taking $c = N_n$ this shows that $e_n(\bar{\mathcal{H}}_{a,b}) \geq 2^{-n(a+b)/2}/L$ so that $2^{n/2} e_n(\bar{\mathcal{H}}) \geq 2^{-n(a+b-1)/2}/L$ and we have proved (F.5). □

A good problem is the converse of Proposition F.0.3. Is it true that $\gamma_{2,2}(\bar{\mathcal{H}}_{a,b}) < \infty$ when $a + b \geq 1$?[3]

Proof of Proposition F.0.2. One proves that for n such that $2^n \leq p/L$ one has

$$2^{n/2} e_n(\mathcal{H}_{a,b}) \geq 2^{2p} 2^{-n(a+b-1)/2}/L \tag{F.6}$$

by a "discrete version" of the previous construction. □

One should note that for $a = b = 0$ (F.6) yields $2^n e_n(\mathcal{H}_{0,0}) \geq 2^{2p} 2^{n/2}/L$, and since this holds for $2^n \geq p/L$ this does not disprove (F.3).

Moral of this appendix: To attack the remaining matching problems, use γ_2 and not $\gamma_{2,2}$.

[3] The conjecture that this holds whenever $a + b \geq 1/2$ stated in page 194 of [153] was certainly too hasty!!

G. Solutions of Selected Exercises.

As the purpose of the exercises is to have the reader (rather than the author) work, the solutions are sketchy, and have not been worked out with the same dedication as the rest of this book. Therefore expect much lousiness and some plain nonsense.

Exercise 1.3.1 Considering for each (s,t) the largest k with $d(s,t) \le 2^{-k}$ yields

$$\sup_{s,t\in G} \frac{|X_s - X_t|^p}{d(s,t)^\beta} \le L \sum_{k\ge 0} \sup_{s,t\in G; d(s,t)\le 2^{-k}} 2^{k\beta} |X_s - X_t|^p \ .$$

Since $\mathsf{E}\sup_{s,t\in G;d(s,t)\le 2^{-k}} |X_s - X_t|^p \le K(m,p,\alpha) 2^{k(m-\alpha)}$ by (1.10) taking expectation yields (1.12) since $\beta + m - \alpha < 0$.

Exercise 1.3.2 By Jensen's inequality we have $\varphi(\mathsf{E}\max_i V_i) \le \mathsf{E}\varphi(\max_i V_i)$. Furthermore $\varphi(\max_i V_i) \le \sum_i \varphi(V_i)$, so that $\mathsf{E}\varphi(\max_i V_i) \le \sum_i \mathsf{E}\varphi(V_i)$.

Exercise 1.3.3 It follows from (1.13) and (1.14) that the r.v. Y_n of (1.5) satisfies $\mathsf{E}Y_n/c_n \le \varphi^{-1}(K(m)2^{nm}d_n)$, and (1.15) follows by combining with (1.7).

Exercise 1.4.3 The distance d associated to Brownian motion is given by $d(s,t) = \sqrt{|s-t|}$ and $N([0,1],d,\epsilon) \le L\epsilon^{-2}$. The condition $|s-t| \le \delta$ implies $d(s,t) \le \sqrt{\delta}$. Dudley's bound is then $L\int_0^{\sqrt{\delta}} \sqrt{\log{(L/\epsilon^2)}}\mathrm{d}\epsilon \le L\sqrt{\delta \log(2/\delta)}$.

Exercise 2.2.2 Just use that $|X_t| \le |X_t - X_{t_0}| + |X_{t_0}| \le \sup_{s,t} |X_s - X_t| + |X_{t_0}|$.

Exercise 2.3.1 Because $\mathsf{P}(Y \ge a\mathsf{E}Y) \le 1/a$ by Markov's inequality.

Exercise 2.3.3 (a) This means that given $L_1 > 0$ there exists L_2 such that $\sup_x xy - L_1 x^3 \le y^{3/2}/L_2$, which is proved by computing this supremum. (b) Let us then assume that $p(u) \le L_1 \exp(-u^2/L_1)$ for $u \ge L_1$. Given a parameter A, for $Au \ge L_1$ we have $p(Au) \le L_1 \exp(-A^2u^2/L_1)$. Also, we have $p(Au) \le 1$, so that $p(Au) \le 2\exp(-u^2)$ for $u \le \sqrt{\log 2}$. Assuming that $A\sqrt{\log 2} \ge L_1$, it suffices that $L_1 \exp(-A^2u^2/L_1) \le 2\exp(-u^2)$ for $u \ge \sqrt{\log 2}$. This is true as soon as $L_1 \le 2\exp(u^2(A/L_1 - 1))$ for $u \ge \sqrt{\log 2}$ and in particular as soon as A is large enough that $L_1 \le 2\exp(\log 2(A^2/L_1 - 1))$. (c) Taking logarithms, it suffices to prove that for $x \ge 0$ and a constant L_1