- In this manner the same representation of $\mathcal{P}^*$ can be described in many different ways, but we do not get confused because we know that all that matters is the restriction of V' to $\mathcal{V}$ and G_{p^*}.

These methods will be illustrated in Section 9.6.

Exercise 9.5.8. Consider the case $m > 0$ and $\mathcal{H}_0 = \mathcal{V} = \mathbb{C}^2$, and denote by U_R the representation constructed in Theorem 9.5.3 with $V(A) = A$, and U_L the representation constructed with $V(A) = A^{\dagger-1}$. Since $A = A^{\dagger-1}$ for $A \in SU(2)$ these representations are unitarily equivalent. The purpose of this exercise is to check this directly. We assume that D_p is Hermitian, so that $\kappa(D_p)$ is a pure boost. Prove that the map W from $\mathcal{H}$ to $\mathcal{H}$ given by $W(\varphi)(p) = D_p^{-2}(\varphi(p))$ satisfies $\|W(\varphi)(p)\|_{L,p} = \|\varphi(p)\|_{R,p}$ and check that $U_L W = W U_R$.

9.6 Fundamental Representations

Given $m \geq 0$ and the point p^* of X_m, considering an irreducible unitary representation V of the little group G_{p^*}, Theorem 9.4.2 produces an irreducible representation of $\mathcal{P}^*$. In this section we name the most important representations of $\mathcal{P}^*$ obtained in this manner, and we show how they can be described within the more friendly setting of Theorem 9.5.3.

9.6.1 Massive Particles

Let us start with the case $m > 0$. Then the little group G_{p^*} of p^* is $SU(2)$, see (9.14). We have constructed the fundamental family π_j of unitary representations of this group in Section 8.2. We will however use the equivalent form (8.14) of these representations using tensor products, which will be recalled in a few lines. The importance of the following representations cannot be overstated.

Definition 9.6.1. For $m > 0$, $j \geq 0$, the representation $\pi_{m,j}$ is the unitary representation of $\mathcal{P}^*$ induced by the representation π_j of the little group $SU(2)$.

It should be stressed that this representation depends on m (as does the point p^*, see (9.13)) despite the fact that the little group does not depend on $m > 0$.

The representation $\pi_{m,0}$ is the representation of Section 4.8. A concrete realization of $\pi_{m,1}$ is given in Proposition 9.5.6, and we can give a concrete realization of $\pi_{m,j}$ for $j > 1$ by a rather straightforward generalization of that construction. We consider the space $\mathcal{V} = \mathcal{H}_0 = \mathcal{S}_j$ of symmetric tensors $x = (x_{n_1,n_2,\ldots,n_j})$ with $n_k \in \{1, 2\}$, and the representation V of $SL(2, \mathbb{C})$ on $\mathcal{S}_j$ given by

$$V(A)(x) = \Big(\sum_{k_1,\ldots,k_j} A_{n_1,k_1} \cdots A_{n_j,k_j} x_{k_1,\ldots,k_j} \Big) . \tag{9.38}$$

It is proved in Proposition 8.3.1 that the restriction of V to $\mathcal{V}$ and the little group

$SU(2)$ is unitarily equivalent to π_j, so that the representation constructed by Theorem 9.5.3 for this choice of V is equivalent to $\pi_{m,j}$. We then have $\mathcal{V}_p = \mathcal{V}$ for each p. According to Lemma 9.5.7 and the footnote on page 235 the norm $\|u\|_p$ is given by $\|u\|_p^2 = (V(M(P(p))/mc)(u), u)$. One may also use the formula of Exercise 9.5.5 for the norm.

9.6.2 Massless Particles

In the remainder of this section we study the case of massless particles, $m = 0$. This case is far more elusive than the case $m > 0$ due to the more complicated nature of the little group. This material will be instrumental in understanding what photons are in Section 9.13, but this in itself is a side story to our main line of investigation, and it may be omitted at first reading.[30] When we work specifically in the case $m = 0$ we will denote by G_0 the little group G_{p^*}, and the first step is to compute this group.[31] We recall that when $m = 0$, $p^* = (1, 0, 0, 1)$ and then the matrix $M(p^*)$ of (8.19) is given by

$$M(p^*) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} . \tag{9.39}$$

The condition that $A \in G_0$, i.e. $\kappa(A)(p^*) = p^*$, that is $AM(p^*)A^\dagger = M(p^*)$, yields in a straightforward manner that A has to be of the type

$$A = \begin{pmatrix} a & b \\ 0 & a^* \end{pmatrix} , \tag{9.40}$$

where $a, b \in \mathbb{C}$ and $aa^* = 1$.

Definition 9.6.2. For $j \in \mathbb{Z}$ and $A \in G_0$ as in (9.40) we define $\hat{\pi}_j(A) = a^j \in \mathbb{C}$.

Please keep in mind that $\hat{\pi}_j(A)$ is a *number*. However, we can think of $\hat{\pi}_j$ as a one-dimensional representation, by the formula $\hat{\pi}_j(A)(x) = \hat{\pi}_j(A)x$.

We recall that for a matrix A, we denote by A^* the conjugate matrix, where every entry has been replaced by its complex conjugate. The following is trivial but useful.

Lemma 9.6.3. *The map $A \to A^*$ is a group automorphism of G_0 and $\hat{\pi}_j(A^*) = \hat{\pi}_{-j}(A)$.*

[30] Generally speaking, all the considerations concerning massless particles constitute a side story and can be omitted at first reading. The reason massless particles are treated in great detail is that the author wanted to truly understand this situation, but could not easily find a satisfactory treatment in the literature.

[31] The reader should observe that there are only two possible values for the little group: G_0 and $SU(2)$.

Definition 9.6.4. Assume $m = 0$. For $j \in \mathbb{Z}$, $j \neq 0$, the representation $\pi_{0,j}$ is the unitary representation of $\mathcal{P}^*$ induced by the representation $\hat{\pi}_j$ of the little group G_0.

The following exercise provides a closer look at the little group G_0, and brings forward the fact that this group is itself a semi-direct product.

Exercise 9.6.5. (a) Consider the group $\mathbb{U}$ of complex numbers of modulus 1. Prove that the operation on $\mathbb{C} \times \mathbb{U}$ defined by

$$(c, a)(c', a') = (c + a^2 c', aa')$$

is a group operation. This group is denoted by G.
(b) Prove that G can be naturally identified as a double cover of the group of transformations of $\mathbb{R}^2$ generated by translations and rotations. Hint: consider the operation of G on $\mathbb{C}$ given by $(c, a) \cdot z = c + a^2 z$. Prove that it is an action, i.e. $(c', a') \cdot ((c, a) \cdot z) = ((c', a')(c, a)) \cdot z$.
(c) Prove that the map which sends the matrix (9.40) to the element (ab, a) of G is a group isomorphism from G_0 to G.
(d) Given $j \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$, prove that the formula

$$U(c, a)(f)(w) = a^j \exp(\mathrm{i}\,\mathrm{Im}(\alpha c^* w)) f(a^{-2} w)$$

(where $\mathrm{Im}\, z$ is the imaginary part of the complex number z) defines a unitary representation of G in the set of square-integrable functions on $\mathbb{C}$. Do you see any connection between this formula and the formulas defining representations of $\mathcal{P}^*$? Why is this the case? Hint: study Section A.5.

One may construct unitary representations of $\mathcal{P}^*$ using the unitary representations of G_0 exhibited in the previous exercise, but they do not correspond to any known particle, and for physics the important case is that of Definition 9.6.4.

We now explore concrete realizations of the representation $\pi_{0,j}$. These realizations are neither very important nor very enlightening, but they do make the point that we are dealing with very non-trivial structures.

Proposition 9.6.6. *For*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \tag{9.41}$$

and $p \in X_0$ *let us define* $\xi(A, p) = d(p^0 + p^3) - b(p^1 + \mathrm{i}p^2)$. *Then given* A, λ_0*-a.e. we have* $\xi(A, p) \neq 0$. *The formula*

$$U(a, A)(\varphi)(p) = \exp(\mathrm{i}(a, p)/\hbar) \Big(\frac{\xi(A, p)}{|\xi(A, p)|} \Big)^{-j} \varphi(A^{-1}(p)) \tag{9.42}$$

defines a unitary representation of $\mathcal{P}^*$ *in* $L^2 := L^2(X_0, \mathrm{d}\lambda_0)$. *This representation is unitarily equivalent to* $\pi_{0,j}$.

One striking feature of the formula is that the crucial factor $(\xi(A,p)/|\xi(A,p)|)$ does not change if we replace p by λp for some $\lambda > 0$.

We are going to show that this formula is simply the formula (9.7) when V is the representation $\hat{\pi}_j$ of G_0 and when we use an adequate choice of D_p.

Proposition 9.6.7. *For $p \in X_0$ we have $p^0 + p^3 \geq 0$. When $p^0 + p^3 \neq 0$ the matrix*

$$D_p = \frac{1}{\sqrt{2(p^0+p^3)}} \begin{pmatrix} p^0+p^3 & 0 \\ p^1 + \mathrm{i}p^2 & 2 \end{pmatrix} \in SL(2,\mathbb{C}) \tag{9.43}$$

satisfies $D_p(p^*) = p$, where $p^* = (1,0,0,1)$.

Proof The first claim follows from the fact that $(p^1)^2 + (p^2)^2 = (p^0)^2 - (p^3)^2$ and $p^0 \geq 0$. When A is as in (9.41) it is straightforward to use the definition $M(A(p^*)) = AM(p^*)A^\dagger$ of $A(p)(= \kappa(A)(p))$ to check that the relation $p := A(p^*)$ amounts to

$$2|a|^2 = p^0 + p^3 \; ; \; 2a^*c = p^1 + \mathrm{i}p^2 \; ; \; 2|c|^2 = p^0 - p^3 \; , \tag{9.44}$$

from which one readily discovers the choice $a = \sqrt{(p^0+p^3)/2}$ and $c = (p^1 + \mathrm{i}p^2)/\sqrt{2(p^0+p^3)}$, the last relation of (9.44) being then a consequence of the relation $(p^1)^2+(p^2)^2 = (p^0)^2-(p^3)^2$. The natural choice $b = 0$ together with the requirement $D_p \in SL(2,\mathbb{C})$ then leads to the formula (9.43). $\square$

Proof of Proposition 9.6.6. We make the choice (9.43) when $p^0 + p^3 \neq 0$ (and any arbitrary choice on the negligible set where $p^3 = -p^0$), and we carry out the computation of the expression (9.7).

The basic observation is that for a matrix $\tilde{A}$ of the type (9.40), if we know that $a = \lambda\alpha$ for some $\alpha \in \mathbb{C}$ and some $\lambda > 0$ then since $|a| = 1$ we have $a = \alpha/|\alpha|$ and therefore $\hat{\pi}_j(\tilde{A}) = (\alpha/|\alpha|)^j$. Recalling that $p^0 + p^3 \geq 0$, D_p^{-1} is a lower-triangular matrix with diagonal coefficients in $\mathbb{R}^+$. When we consider a matrix $B = (B_{i,j})$ with $\tilde{A} := D_p^{-1}B \in G_0$ it follows that $\tilde{A}_{1,1} = \lambda B_{1,1}$ with $\lambda > 0$. Consequently $\hat{\pi}_j(D_p^{-1}B) = \tilde{A}_{1,1}^j = (B_{1,1}/|B_{1,1}|)^j$. Consider now $A \in SL(2,\mathbb{C})$. Then $\hat{\pi}_j(D_p^{-1}AD_{A^{-1}(p)}) = (B_{1,1}/|B_{1,1}|)^j$ where $B = AD_{A^{-1}(p)}$. From (9.43) the first column of D_p is the same as λ times the first column of $M(p)$ for a certain $\lambda > 0$. Thus the first column of $D_{A^{-1}(p)}$ is the same as the λ times the first column of $M(A^{-1}(p))$ for a certain $\lambda > 0$. Consequently $B_{1,1}/|B_{1,1}| = C_{1,1}/|C_{1,1}|$ where $C = AM(A^{-1}(p))$. But then $M(A^{-1}(p)) = A^{-1}M(p)(A^{-1})^\dagger$ so that $C = M(p)(A^{-1})^\dagger$. A straightforward calculation yields $C_{1,1} = \xi(A,p)^*$. Finally $(\xi(A,p)^*/|\xi(A,p)|)^j = (\xi(A,p)/|\xi(A,p)|)^{-j}$. $\square$

This near-miraculous calculation does not shed much light on why the formula (9.42) defines a representation. The following exercise will provide a direct proof of this fact.

Exercise 9.6.8. (a) Prove that the formula

$$U(a,A)(\varphi)(p) = \exp(\mathrm{i}(a,p)/\hbar)g(A,p)\varphi(A^{-1}(p))$$

defines a representation of $\mathcal{P}^*$ if for each $A, B \in SL(2,\mathbb{C})$ and each $p \in X_0$ we have

$$g(B,p)g(A,B^{-1}(p)) = g(BA,p) \ . \tag{9.45}$$

In the sequel, please do not worry about the rare cases where the definitions make no sense.
(b) For $z \in \mathbb{C}$ and $A \in SL(2,\mathbb{C})$ as in (9.41) define $A \cdot z = (c+dz)/(a+bz)$. Prove that this defines an action of $SL(2,\mathbb{C})$ on $\mathbb{C}$, i.e. $A \cdot (B \cdot z) = (AB) \cdot z$.
(c) For $A \in SL(2,\mathbb{C})$ and $z \in \mathbb{C}$ define $f(A,z) = d - bz$. Prove that

$$f(B,z)f(A,B^{-1} \cdot z) = f(BA,z) \ . \tag{9.46}$$

(d) What is the relation with Proposition 9.6.6? Hint: find a suitable identification of the rays of X_0 with $\mathbb{C}$, which transforms the natural action of $SL(2,\mathbb{C})$ on these rays into the previous one.

Exercise 9.6.9. Think of $v \in \mathbb{C}^2$ as a column matrix. We denote by v_1, v_2 the components of $v \in \mathbb{C}^2$. We recall the action of matrix multiplication Av for $A \in SL(2,\mathbb{C})$ and $v \in \mathbb{C}^2$.
(a) Prove that for $v \in \mathbb{C}^2$ there exists a unique $p(v) \in X_0$ such that $M(p(v)) = vv^\dagger$.
(b) Consider $j \in \mathbb{Z}$ and the space $\mathcal{H}_j \subset L^2(\mathbb{C}^2)$ consisting of functions f such that for any complex number θ of modulus 1 we have $f(\theta v) = \theta^{-j} f(v)$. Prove that the formula

$$V(a,A)(f)(v) = \exp(\mathrm{i}(a,p(v))/\hbar)f(A^{-1}v) \tag{9.47}$$

defines a unitary representation of $\mathcal{P}^*$ in $\mathcal{H}_j$.
(c) For $p \in X_0$ define (whenever possible) $w(p) \in \mathbb{C}^2$ by $\sqrt{p^0+p^3}\,w(p) = \begin{pmatrix} p^0+p^3 \\ p^1+\mathrm{i}p^2 \end{pmatrix}$. Prove that $p(w(p)) = p$. Prove that if $vv^\dagger = M(p)$ then $v = (v_1/|v_1|)w(p)$.
(d) Prove that for $B \in SL(2,\mathbb{C})$ we have $Bw(p) = (Bw(p)_1/|Bw(p)_1|)w(Bp)$.
(e) Prove that the representations (9.42) and (9.47) are unitarily equivalent. Hint: Use the map $T : \mathcal{H}_j \to L^2(X_0, \mathrm{d}\lambda_0)$ given by $T(f)(p) = f(w(p))$.

In the remainder of this section we study the description of the representation $\pi_{0,j}$ of $\mathcal{P}^*$ within the framework of Theorem 9.5.3, using this theorem for $m = 0$. We assume $j \geq 1$. We consider again the space $\mathcal{H}_0 = \mathcal{S}_j$ of symmetric tensors and the representation V of (9.38). In the important case $j = 1$ we then have $\mathcal{H}_0 = \mathbb{C}^2$ and V is the action on $\mathbb{C}^2$ by matrix multiplication, $V(A)(x) = A(x)$. Consider the tensor $g \in \mathcal{S}_j$ given by $g_{n_1,\ldots,n_j} = 0$ unless all indices are equal to 1, in which case $g_{n_1,\ldots,n_j} = 1$. Thus, when $j = 1$, g is the vector with $g_1 = 1$ and $g_2 = 0$. We denote by $\mathcal{V}$ the linear span of g, which is then a space of dimension 1. For $A \in G_0$ given by (9.40) we have $V(A)(g) = a^j g = \hat{\pi}_j(A)g$. Thus $\mathcal{V}$ is invariant under $V(A)$, and the restriction of V to G_0 and $\mathcal{V}$ is unitarily equavalent to the representation $\hat{\pi}_j$ of G_0. Consequently, the representation of $\mathcal{P}^*$ constructed by Theorem 9.5.3 is unitarily equivalent to the representation $\pi_{0,j}$.

Let us now compute the norm $\|u\|_p$ of an element u of $\mathcal{V}_p$. Such an element is of the type

$$u = V(A)(\lambda g) = (\lambda A_{n_1,1} A_{n_2,1} \cdots A_{n_j,1}) \ ,$$

and $p = A(p^*) = \kappa(A)(p^*)$ i.e. $AM(p^*)A^\dagger = M(p)$. Using (9.39), it is straightforward to see that if A is as in (9.41) then $p^0 = |a|^2 + |c|^2$. Thus

$$\|u\|_p^2 = \|V(A)^{-1}(u)\|^2 = \|\lambda g\|^2 = |\lambda|^2$$

whereas since $|A_{1,1}|^2 + |A_{2,1}|^2 = |a|^2 + |c|^2$,

$$\|u\|^2 = |\lambda|^2 \sum_{n_1,\ldots,n_j} |A_{n_1,1}|^2 \cdots |A_{n_j,1}|^2 = |\lambda|^2 (|a|^2 + |c|^2)^j = (p^0)^j |\lambda|^2 \ .$$

Consequently $\|u\|_p^2 = \|u\|^2/(p^0)^j$. We have proved the following.

Proposition 9.6.10. *For $j \geq 1$ consider the space $\mathcal{H}$ of functions $\varphi : X_0 \to \mathcal{S}_j$ which satisfy $\varphi(p) \in \mathcal{V}_p$ for each p, provided with the norm*

$$\|\varphi\|^2 = \int \mathrm{d}\lambda_0(p) \frac{\|\varphi(p)\|^2}{(p^0)^j} \ . \tag{9.48}$$

Then the formula

$$U(a, A)(\varphi)(p) = \exp(\mathrm{i}(a, p)/\hbar) V(A)[\varphi(A^{-1}(p))] \tag{9.49}$$

defines a unitary representation of $\mathcal{P}^*$ on $\mathcal{H}$, which is unitarily equivalent to the representation $\pi_{0,j}$.

Exercise 9.6.11. Find the corresponding statement for the representation $\pi_{0,j}$.

When $j = 1$ our next result describes the space $\mathcal{V}_p$ by an explicit formula.

Lemma 9.6.12. *When $j = 1$ we have*

$$\mathcal{V}_p = \{u \in \mathcal{H}_0 = \mathbb{C}^2 \ ; \ M(Pp)(u) = 0\}. \tag{9.50}$$

Proof We recall that when $j = 1$, we have $\mathcal{H}_0 = \mathbb{C}^2$ and $g \in \mathbb{C}^2$ is the vector whose components are $g_1 = 1$ and $g_2 = 0$. Its linear span $\mathcal{V}$ is the set of vectors which have a second component equal to zero. Thus (9.50) holds for $p = p^*$ since, as in (9.39), we have

$$M(Pp^*) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \ .$$

For the general case, we observe that since $PA(p^*) = P\kappa(A)(p^*) = \kappa(A^{\dagger -1})(Pp^*)$ by (8.36), it holds that

$$M(PA(p^*)) = A^{\dagger -1} M(Pp^*) A^{-1} \; . \tag{9.51}$$

If now $v \in \mathcal{V}_p$ we have $v = D_p(u)$ for a certain $u \in \mathcal{V}$. Applying the previous relation to v and taking $A = D_p$ shows that $M(Pp)(v) = 0$ for $v \in \mathcal{V}_p$. Since $\mathcal{V}_p$ has dimension 1, this proves (9.50). □

Therefore the space $\mathcal{H}$ has a very clean description as the set of functions φ from X_0 to $\mathbb{C}^2$ which satisfy the equation $M(Pp)(\varphi(p)) = 0$. Let us state again this result.

Proposition 9.6.13. *Consider the space $\mathcal{H}$ of functions $\varphi : X_0 \to \mathbb{C}^2$ which satisfy the equation $M(Pp)(\varphi(p)) = 0$, provided with the norm (9.48). Then the formula*

$$U(a, A)(\varphi)(p) = \exp(\mathrm{i}(a, p)/\hbar) A[\varphi(A^{-1}(p))] \tag{9.52}$$

defines a unitary representation of $\mathcal{P}^*$ on $\mathcal{H}$, which is unitarily equivalent to the representation $\pi_{0,-1}$.

Recalling that $M(p) = p^\mu \sigma_\mu$ we can consider the matrices $\bar{\sigma}_\mu$ such that $M(Pp) = p^\mu \bar{\sigma}_\mu$. That is $\bar{\sigma}_0 = \sigma_0$ and $\bar{\sigma}_i = -\sigma_i$ for $1 \le i \le 3$.[32] The equation $M(Pp)(\varphi(p)) = 0$ is then written as $p^\mu \bar{\sigma}_\mu v(p) = 0$ in physics books.

Exercise 9.6.14. Find the corresponding statement for the representation $\pi_{0,1}$.

Exercise 9.6.15. In the case of a general value of $j \ge 1$ prove that $u \in \mathcal{V}_p$ if and only if $(D_p D_p^\dagger M(Pp) \otimes I \otimes \cdots \otimes I)(u) = 0$.

9.7 Particles, Spin, Representations

The present section is a continuation of Section 4.9, which the reader may like to review now. What are the properties of a particle to which corresponds the representation $\pi_{m,j}$ or $\pi_{0,j}$? Let us first consider the case of $\pi_{m,j}$ where $m > 0$. As on page 132 we may argue that *the particle is of mass m* and here we investigate its spin. We certainly expect from Section 9.1 that

The representation $\pi_{m,j}$ of Definition 9.6.1 describes a particle of spin $j/2$. (9.53)

In Definition 8.7.1 we defined what is a particle of spin $j/2$ in Non-Relativistic Quantum Mechanics. This theory is however not a part of Quantum Field Theory, just like, say, Newton's theory of gravitation is not a part of General Relativity. So we cannot check that we are here in the setting of Definition 8.7.1. The best

[32] So that, indeed, $\bar{\sigma}_\mu = \sigma^\mu$. The point of the new notation is that the formalism forbids the expression $p^\mu \sigma^\mu$.

we can do is to decide that we will *define* the spin of a particle described by the representation $\pi_{m,j}$ as being $j/2$ and to convince ourselves that this is reasonably consistent with Definition 8.7.1. For this let us denote by $D_p \in SL(2,\mathbb{C})$ the unique positive definite Hermitian matrix with $D_p(p^*) = (\kappa(D_p)(p^*)) = p$ as provided by Lemma 8.4.6.

Lemma 9.7.1. *We have $D_p^{-1} A D_{A^{-1}(p)} = A$ for each $A \in SU(2)$ and each $p \in X_m$.*

Proof The matrix $C := A^{-1} D_p A = A^\dagger D_p A$ is positive Hermitian. Since $A \in SU(2)$ it holds that $A(p^*) = \kappa(A)(p^*) = p^*$ so that $C(p^*) = A^{-1} D_p A(p^*) = A^{-1}(p)$. Thus C equals $D_{A^{-1}(p)}$, the unique positive Hermitian matrix with this property. □

Consequently, for $A \in SU(2)$, (9.7) yields

$$\tilde{U}(0, A)(\varphi)(p) = V(A)(\varphi(A^{-1}(p))) \, . \tag{9.54}$$

To understand what this means we compare it with (8.34), which we write

$$(U \otimes \pi_j)(A)(\varphi)(\boldsymbol{p}) = \pi_j(A)(\varphi(A^{-1}(\boldsymbol{p}))) \, , \tag{9.55}$$

since $A^{-1}(\boldsymbol{p})$ is the notation we now use for $\kappa(A^{-1})(\boldsymbol{p})$. The representation $\pi_{m,j}$ corresponds to (9.54) with $V = \pi_j$. Then the overwhelming analogy between (9.54) and (9.55) supports (9.53).[33]

The heuristic approach of Section 9.1 effortlessly discovers the method of induced representations (9.6). That is, *assuming that the particle is characterized by its spin and its mass*, it discovers how the Poincaré group acts on it. On the other hand, Wigner *proved* that all the projective representations of $\mathcal{P}^*$ are obtained as induced representations. This goes much further: the way the Poincaré group acts on a particle of given positive mass depends only on its spin. It *proves* that there is no other possible characteristic of the particle involved there.

Next we turn to the case of the representation $\pi_{0,j}$ of Definition 9.6.4, acting as in this definition on the space $L^2(X_0, \lambda_0)$. As on page 132 we argue that a particle described by this representation must be massless.

What property of this particle is reflected by the representation $\hat{\pi}_j$ of the little group? In imprecise but picturesque terms, we will show that

$$\begin{gathered}\text{a rotation of angle } \theta \text{ around the direction of motion } \boldsymbol{p} \\ \text{multiplies the state by a phase } \exp(-\mathrm{i}j\theta/2).\end{gathered} \tag{9.56}$$

Keeping in mind the result of Exercise 8.8.2 we describe the situation by saying: "the particle has helicity $j/2$".[34] We will think of a particle with helicity $j/2$ as

[33] None of the arguments above pretends to be a rigorous deduction of anything.

[34] It is the desire to have $\pi_{0,j}$ describe particles of helicity $j/2$ (rather than $-j/2$) which dictates the choice $\hat{\pi}_j(A) = a^{-j}$ as opposed to the seemingly more natural choice $\hat{\pi}_j(A) = a^j$.

having spin $|j|/2$. Thus a massless particle of given spin $j/2$ ($j > 0$) comes in two versions, with helicity $\pm j/2$. In contrast, a massive particle is determined by its mass and its spin.

The similarity of terminology should not hide that the massive and massless cases are very different.

Exercise 9.7.2. What differences do you see?

To prove (9.56), let us first consider the ideal case where $\varphi(p) = 0$ unless $p = \lambda p^* = (\lambda, 0, 0, \lambda)$ for some $\lambda > 0$. The direction of motion is then the z-axis. The element $A = \exp(-\mathrm{i}\theta\sigma_3/2) = \begin{pmatrix} \exp(-\mathrm{i}\theta/2) & 0 \\ 0 & \exp(\mathrm{i}\theta/2) \end{pmatrix}$ is such that $\kappa(A)$ corresponds to a rotation of angle θ around the z-axis. The quantity $\xi(A, p)$ of Proposition 9.6.6 equals $2\lambda \exp(\mathrm{i}\theta/2)$ and the quantity $\xi(A,p)/|\xi(A,p)|$ then equals $\exp(\mathrm{i}\theta/2)$. Thus (9.42) takes the form $U(0, A)(\varphi)(p) = \exp(\mathrm{i}j\theta/2)\varphi(p)$ which proves the claim (9.56).

To prove (9.56) in the general case, the basic observation is that if R is a rotation of angle θ around the direction of a vector $\boldsymbol{u}$, and S is another rotation, then SRS^{-1} is a rotation of angle θ around the direction of $S(\boldsymbol{u})$. Considering a rotation $\kappa(U_p)$ which sends $(p^0, 0, 0, |\boldsymbol{p}|)$ to $p = (p^0, \boldsymbol{p})$, and

$$A = U_p \exp(-\mathrm{i}\theta\sigma_3/2) U_p^{-1} , \tag{9.57}$$

then $\kappa(A)$ is a rotation of angle θ around the direction of $\boldsymbol{p}$, and

$$U(0, A)(\varphi) = U(0, U_p) U(0, \exp(-\mathrm{i}\theta\sigma_3/2)) U(0, U_p^{-1})(\varphi) . \tag{9.58}$$

In the ideal case where $\varphi(p')$ is $\neq 0$ only when $\boldsymbol{p}'$ is in the direction of $\boldsymbol{p}$, then (9.42) shows that $\psi := U(0, U_p^{-1})(\varphi)$ is such that $\psi(p') \neq 0$ only for $\boldsymbol{p}'$ in the direction of e_3. We have then shown that $U(0, \exp(-\mathrm{i}\theta\sigma_3/2))(\psi) = \exp(-\mathrm{i}j\theta/2)\psi$ and (9.58) implies as desired that $U(0, A)(\varphi) = \exp(-\mathrm{i}j\theta/2)\varphi$.

Exercise 9.7.3. The purpose of the present challenging exercise is to try to look at the representation $\pi_{0,j}$ using physicists' tools. We denote by $R(\theta, \boldsymbol{u})$ the rotation of angle θ and axis $\boldsymbol{u}$, where $\boldsymbol{u}$ is a unit vector. Considering a representation W of $SO(3)$ (which models the action of rotations) one defines a self-adjoint operator $J_{\boldsymbol{u}}$ by

$$J_{\boldsymbol{u}} = -\lim_{\theta \to 0} \frac{\hbar}{\mathrm{i}\theta}(W(R(\theta, \boldsymbol{u})) - 1) . \tag{9.59}$$

It is called the *angular momentum with respect to axis* $\boldsymbol{u}$.
(a) When we consider instead a representation of $SU(2)$, convince yourself after studying the first three sections of Appendix D that instead of (9.59) one should use the formula

$$J_{\boldsymbol{u}} = -\lim_{\theta \to 0} \frac{\hbar}{\mathrm{i}\theta}(W(\exp(-\mathrm{i}\theta \boldsymbol{u} \cdot \boldsymbol{\sigma}/2)) - 1) , \tag{9.60}$$

where the notations are those of Appendix D.

We first assume $j = 1$ and consider the realization of $\pi_{0,1}$ described in Proposition 9.6.13. In this case the representation W modeling the action of rotations is given by $W(A) = U(0, A)$, where $U(a, A)$ is defined in (9.52).

(b) In the ideal case where φ is non-zero only for a given value of $p = (p^0, \boldsymbol{p})$ and where the unit vector $\boldsymbol{u}$ is $\boldsymbol{p}/|\boldsymbol{p}|$ show that $J_{\boldsymbol{u}}(\varphi)$ is non-zero only at p and that

$$J_{\boldsymbol{u}}(\varphi)(p) = -\frac{\hbar}{2}\boldsymbol{u} \cdot \boldsymbol{\sigma}(\varphi(p)) \ . \tag{9.61}$$

(c) Use the condition $M(Pp)(\varphi(p)) = 0$ to show that the previous relation implies

$$J_{\boldsymbol{u}}(\varphi)(p) = -\frac{\hbar}{2}\varphi(p) \ ,$$

and interpret this relation as "the spin in the direction of motion is $-1/2$".

(d) Generalize the previous considerations to the case of an arbitrary value of j.

Physicists find the considerations of the previous two pages as self-evident, and write the following argument: "since $-\hbar\boldsymbol{u}\cdot\boldsymbol{\sigma}/2$ measures the angular momentum in the direction of the unit vector $\boldsymbol{u}$, the operator $-\hbar\boldsymbol{p}\cdot\boldsymbol{\sigma}/(2|\boldsymbol{p}|)$ measures the angular momentum in the direction of motion, but according the equation $M(Pp)(\varphi(p)) = 0$, this operator is simply multiplication by $-\hbar/2$."[35]

Let us then summarize the situation:

- For $m > 0$, the representation $\pi_{m,j}$ corresponds to a particle of mass m and spin $j/2$.
- The representation $\pi_{0,j}$ corresponds to a massless particle of spin $|j|/2$ and helicity $j/2$.

The mathematical subtleties of the treatment of massless particles are largely irrelevant for physics.[36] It is possible for physics to assume that every particle has a tiny mass. The results of every experiment conceivable today would be the same if photons had a rest mass of 10^{-1000} kg, and articles finding experimental upper bounds for this rest mass do get published.[37] Still, as the mathematics are interesting, starting with Section 9.12 we will indulge for a few pages in thinking about massless particles.

[35] I do not see how to give a precise meaning to this without the entering considerations of the previous exercise. If you think I am nit picking ask yourself what is the state space on which these operators act.

[36] As of today, the photon is the only confirmed massless particle. The hypothetical graviton would also be massless, but with spin 2. Neutrinos were long thought to be be massless, but now are believed to have a very small positive mass.

[37] Try the search for the words "photon mass limit"! The upper bound 10^{-16}e.V. seems very solid.

9.8 Abstract Presentation of Induced Representations

In the present section we give a more abstract, but more intrinsic description of the previous representations. Using it certain results become far simpler. We will witness this when proving Proposition 9.4.6 below, again in Section 9.12, and crucially in Section 10.4. The method is a special case of a very general method in representation theory, the so-called "method of induced representations". In the setting of representations of a semi-direct products $N \rtimes H$ (with N abelian) the key idea of the method can be implemented in a simple way.[38]

We carry out the method only in the case of the physically relevant representations of the Poincaré group.[39] We start with a fixed point $p^* \in X_m$ ($m \geq 0$), with little group $G_{p^*} = \{A \in SL(2,\mathbb{C}); A(p^*) = p^*\}$ for the action of $SL(2,\mathbb{C})$. For each $p \in X_m$ we fix an element $D_p \in SL(2,\mathbb{C})$ with $D_p(p^*) = p$. Consider a unitary representation V of G_{p^*} in a finite-dimensional space $\mathcal{V}$, and the induced representation $\tilde{U}(a,A)$ of Theorem 9.4.2. It acts on the space $L^2 = L^2(X_m, \mathcal{V}, \mathrm{d}\lambda_m)$. The magic idea (which will transport the representation to something much nicer) is to associate to each function $\varphi \in L^2$ a function $T(\varphi) : SL(2,\mathbb{C}) \to \mathcal{V}$ by the formula

$$T(\varphi)(A) = V(A^{-1}D_{A(p^*)})\varphi(A(p^*)) \ . \tag{9.62}$$

This makes sense because $A^{-1}D_{A(p^*)}(p^*) = A^{-1}A(p^*) = p^*$ so that $A^{-1}D_{A(p^*)} \in G_{p^*}$. Recalling the formula (9.7), let $\psi := \tilde{U}(a,A)(\varphi)$ so that

$$\psi(p) = \tilde{U}(a,A)(\varphi)(p) = \exp(\mathrm{i}(a,p)/\hbar)V(D_p^{-1}AD_{A^{-1}(p)})[\varphi(A^{-1}(p))] \ .$$

For $B \in SL(2,\mathbb{C})$ we compute

$$\begin{aligned} T(\tilde{U}(a,A)(\varphi))(B) &= T(\psi)(B) = V(B^{-1}D_{B(p^*)})[\psi(B(p^*))] \\ &= \exp(\mathrm{i}(a,B(p^*))/\hbar)V(B^{-1}D_{B(p^*)})V(D^{-1}_{B(p^*)}AD_{A^{-1}B(p^*)})[\varphi(A^{-1}B(p^*))] \\ &= \exp(\mathrm{i}(a,B(p^*))/\hbar)V(B^{-1}AD_{A^{-1}B(p^*)})[\varphi(A^{-1}B(p^*))] \\ &= \exp(\mathrm{i}(a,B(p^*))/\hbar)T(\varphi)(A^{-1}B) \ . \end{aligned} \tag{9.63}$$

For a function $f : SL(2,\mathbb{C}) \to \mathcal{V}$ let us then define

$$\pi(a,A)(f)(B) = \exp(\mathrm{i}(a,B(p^*))/\hbar)f(A^{-1}B) \ , \tag{9.64}$$

so that (9.63) means

$$T\tilde{U}(a,A) = \pi(a,A)T \ . \tag{9.65}$$

Thus, if we transport the representation $\tilde{U}(a,A)$ by T it is given by the much simpler formula (9.64). The function $f = T(\varphi)$ of (9.62) satisfies

$$A \in SL(2,\mathbb{C}), C \in G_{p^*} \Rightarrow f(AC) = V(C^{-1})f(A) \ , \tag{9.66}$$

[38] It was a character-forming experience to figure this out.

[39] If it helps you to think in more abstract and general terms, a general scheme to construct unitary representations of semi-direct products is presented in the form of a sequence of exercises at the end of the section.

since $C(p^*) = p^*$ and $V(C^{-1}A^{-1}D_{A(p^*)}) = V(C^{-1})V(A^{-1}D_{A(p^*)})$. Conversely, if the function f satisfies (9.66), then $f = T(\varphi)$ where the function φ is given by $\varphi(p) = f(D_p)$. Indeed,

$$T(\varphi)(A) = V(A^{-1}D_{A(p^*)})f(D_{A(p^*)}) = f(A)\ ,$$

using (9.66) for $C = D_{A(p^*)}^{-1}A$.

Theorem 9.8.1. *Given a unitary representation V of the little group G_{p^*} on a space $\mathcal{V}$, consider the space $\mathcal{F}$ of functions $f : SL(2,\mathbb{C}) \to \mathcal{V}$ which satisfy the condition (9.66), provided with the norm*

$$\|f\|^2 = \int \mathrm{d}\lambda_m(p)\|f(D_p)\|^2\ . \tag{9.67}$$

The formula (9.64) defines a unitary representation of $\mathcal{P}^*$ on $\mathcal{F}$. This representation is unitarily equivalent to the representation induced by V.

Proof We have shown that T is unitary from L^2 to $\mathcal{F}$, so that (9.65) proves the theorem. □

Exercise 9.8.2. Prove directly that the formula (9.64) defines a unitary representation on $\mathcal{F}$. Prove that the norm (9.67) is independent of the choice of $D_p \in SL(2,\mathbb{C})$ with $D_p(p^*) = p$.

Exercise 9.8.3. Following the method of Exercise 9.5.5 prove that when $m > 0$ there is a left-invariant measure $\mathrm{d}\mu$ on $SL(2,\mathbb{C})$ such that for $f \in \mathcal{F}$ we have $\|f\|^2 = \int \mathrm{d}\mu(A)\|f(A)\|^2$.

Exercise 9.8.4. Find a proof of Proposition 9.4.5 using the presentation of the present section.

We are now ready to prove that the method of induced representation constructs irreducible representations.

Proof of Proposition 9.4.6. It suffices to prove the corresponding result for the representations of Theorem 9.8.1. The "only if" part is obvious, since if $\mathcal{W}$ is a subspace of $\mathcal{V}$ which is invariant under V, the space of $\mathcal{W}$-valued functions is invariant under each $\pi(a, A)$.

The non-trivial proof of the converse may be skipped at first reading. We denote by g a non-zero element of $\mathcal{F}$ and by $\langle\cdot,\cdot\rangle$ the inner product in $\mathcal{V}$. Consider an element $f \in \mathcal{F}$ and assume that f is orthogonal to all the elements $\pi(a, A)(g)$. We have to prove that $f = 0$ in the space of functions provided with the norm (9.67), that is we have to prove that $f(D_p) = 0$ $\mathrm{d}\lambda_m$-a.e. The inner product in $\mathcal{F}$ is given by $(f, g) = \int \mathrm{d}\lambda_m(p)\langle f(D_p), g(D_p)\rangle$ so that our hypothesis is that for each value of a, A the integral $\int \mathrm{d}\lambda_m(p)\langle f(D_p), \exp(\mathrm{i}(a, D_p(p^*))/\hbar)g(A^{-1}D_p)\rangle$ is zero. Since $D_p(p^*) = p$ we prove as in Proposition 4.8.4 that for each A in $SL(2,\mathbb{C})$ the function $p \mapsto \langle f(D_p), g(A^{-1}D_p)\rangle$ has to be zero $\mathrm{d}\lambda_m$-a.e. Changing A into A^{-1}, we have proved that for each $A \in SL(2,\mathbb{C})$ the function $p \mapsto \langle f(D_p), g(AD_p)\rangle$ has to be zero $\mathrm{d}\lambda_m$-a.e.

We now use that on the locally compact group $SL(2,\mathbb{C})$ there is a right-invariant measure $\mathrm{d}\mu$.[40] That is, $\mathrm{d}\mu$ is invariant by the transformations $A \mapsto AB$ for $B \in SL(2,\mathbb{C})$. In particular if a function $A \mapsto h(A)$ is zero $\mathrm{d}\mu$-a.e. then for each $B \in SL(2,\mathbb{C})$ the function $A \mapsto h(AB)$ is zero $\mathrm{d}\mu$-a.e. (and conversely). Using Fubini's theorem for the measure $\mathrm{d}\lambda_m \otimes \mathrm{d}\mu$, for $\mathrm{d}\lambda_m$-almost each p the function $A \mapsto \langle f(D_p), g(AD_p)\rangle$ has to be zero $\mathrm{d}\mu$-a.e. Let us fix now such a value of p with the goal of showing that $f(D_p) = 0$ (which will conclude the proof). By right-invariance of μ the function $A \mapsto \langle f(D_p), g(A)\rangle$ is zero $\mathrm{d}\mu$-a.e. and hence, again by right-invariance, for every $C \in G_{p^*}$ the function $A \mapsto \langle f(D_p), g(AC)\rangle = \langle f(D_p), V(C^{-1})g(A)\rangle$ is zero $\mathrm{d}\mu$-a.e.

Denoting by $\mathrm{d}\gamma$ the Haar measure of G_{p^*} and using Fubini's theorem again, for $\mathrm{d}\mu$-almost all values of A the function $C \mapsto \langle f(D_p), V(C^{-1})g(A)\rangle$ is zero $\mathrm{d}\gamma$-a.e., and hence everywhere as this function is continuous. In particular this occurs for a value A for which $g(A) \neq 0$. But since V is irreducible, this proves as desired that $f(D_p) = 0$. □

In the following exercises, we sketch a general method to construct unitary representations of a semi-direct product $N \rtimes H$ where N is commutative and N and H are countable.[41]

Exercise 9.8.5. Consider a countable group G and a subgroup H. The relation $A\mathcal{R}B$ iff $B^{-1}A \in H$ is an equivalence relation on G. The quotient of G by this equivalence relation is denoted by G/H. Describe a natural action of G on G/H, and consider a positive measure λ on G/H which is invariant under this action. Let us assume that we are given a unitary representation V of H in a Hilbert space $\mathcal{V}$. If a function f from G to $\mathcal{V}$ satisfies

$$A \in G, C \in H \Rightarrow f(AC) = V(C^{-1})f(A) , \tag{9.68}$$

prove that it makes sense to define

$$\|f\|^2 = \int_{G/H} \mathrm{d}\lambda(p)\|f(D_p)\|^2$$

where $D_p \in G$ is such that its class in G/H is p. Consider the Hilbert space $\mathcal{H}$ of these functions for which $\|f\| < \infty$. Prove that we may define a unitary representation π of G in $\mathcal{H}$ by the formula $\pi(A)(f)(B) = f(A^{-1}B)$.

Exercise 9.8.6. This exercise continues the previous one. We consider a countable

[40] On a locally compact group there exist both right-invariant and left-invariant measures, which may be different, although it can be shown that this does not happen in the case of $SL(2,\mathbb{C})$. A left-invariant measure is constructed in Exercise 9.5.5.

[41] If you know about locally compact groups, you may assume that these groups are locally compact, adding the proper continuity assumptions whenever required.

Abelian group N and a semi-direct product $N \rtimes H$. The law of this group is given by $(a, A)(b, B) = (a + \kappa(A)(b), AB)$ where $\kappa(AB) = \kappa(A)\kappa(B)$ and $\kappa(A)(a + b) = \kappa(A)(a) + \kappa(A)(b)$. A *character* of N is a unitary representation of N on $\mathbb{C}$, and $\hat{N}$ denotes the set of these representations, called the *dual* of N. Prove that we can define an action $\hat{\kappa}$ of H on $\hat{N}$ by $\hat{\kappa}(A)(w)(x) = w(\kappa(A^{-1})(x))$ for $w \in \hat{N}$ and $x \in H$. Let us then fix $w \in \hat{N}$, and define the little group H_w of w as the set of $A \in H$ for which $\hat{\kappa}(A)(w) = w$. Explain why the orbit of w under the action of H identifies with the quotient H/H_w. Consider a unitary representation V of H_w in a Hilbert space $\mathcal{V}$ and the space $\mathcal{H}$ of functions from H to $\mathcal{V}$ defined in the previous exercise. Prove that we define a unitary representation π of $N \rtimes H$ by the formula

$$\pi(a, A)f(B) = \hat{\kappa}(B)(w)(a)f(A^{-1}B) \; .$$

Relate this to formula (9.64).

9.9 Particles and Parity

Now, what about parity? We can act on a particle by translations, rotations, even (in principle) Lorentz transformations. There is however no machine which takes a particle and produces a mirror image of it. Still, many situations are mirror images of each other, and to account for this we should try to build models which involve a parity operator. The first step in this direction is to enlarge the group $\mathcal{P}^*$ by "adding a parity element". This is done simply by considering the group $\mathcal{P}^{*+} = \mathbb{R}^{1,3} \rtimes SL^+(2, \mathbb{C})$ where the group $SL^+(2, \mathbb{C})$ is as in Definition 8.9.3. That is, one adds a new element P' to $SL(2, \mathbb{C})$, so the group $SL^+(2, \mathbb{C})$ consists of the elements $P', A, P'A$ where $A \in SL(2, \mathbb{C})$ with the multiplication rule (8.41): $P'I = P'$, $P'AP' = A^{\dagger -1}$, and κ is extended to $SL^+(2, \mathbb{C})$ by (8.42), i.e. $\kappa(P'A) = P\kappa(A)$, where P is the parity operator $(p^0, \boldsymbol{p}) \to (p^0, -\boldsymbol{p})$. The "parity element" of $\mathcal{P}^{*+}$ is then $(0, P')$.

The obvious question is then how to construct meaningful representations of $\mathcal{P}^{*+}$.

The mass shell X_m is still an orbit under the action of $\mathcal{P}^{*+}$ and Theorem 9.4.2 extends in a straightforward manner to the setting where $SL(2, \mathbb{C})$ is replaced by $SL^+(2, \mathbb{C})$ and $\mathcal{P}^*$ is replaced by $\mathcal{P}^{*+}$. This extension will be called "Extended Theorem 9.4.2".

Let us first say a few words about the really easy case $m > 0$ and $p^* = (mc, 0, 0, 0)$. Then the little group of $SL^+(2, \mathbb{C})$ consists of the group $SU^+(2)$ generated by $SU(2)$ and P', and the element P' commutes with every element of $SU^+(2)$.

Let us compare the irreducible representations of $SU(2)$ and $SU^+(2)$. Consider first an irreducible representation V of $SU^+(2)$. Since P' commutes with every element of $SU^+(2)$ then $V(P')$ commutes with every operator $V(A)$. Since V is irreducible, $V(P')$ is a multiple of the identity $\mathbf{1}$ by Schur's lemma (Lemma 4.5.7). Since P'^2 is the identity, there are two cases: either $V(P')$ is the identity or it is minus the identity. Conversely an irreducible representation of $SU(2)$ can be