If all mathematics were to disappear, physics would be set back exactly one week.

Richard Feynman

Physics should be made as simple as possible, but not simpler.

Albert Einstein

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.

Sydney Coleman

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Introduction

As a teenager in the sixties reading scientific magazines, countless articles alerted me to "the infinities plaguing the theory of Quantum Mechanics." Reaching 60 after a busy mathematician's life, I decided that it was now or never for me to really understand the subject.[1] The project started for my own enjoyment, before turning into the hardest of my scientific life. I faced many difficulties, the most important being the lack of a suitable introductory text. These notes try to mend that issue.

I knew no physics to speak of, but it was not particularly difficult to get a basic grasp of topics such as Classical Mechanics, Electromagnetism, Special and even General Relativity. They are friendly for mathematicians as they can be made rigorous to our liking.

Quantum Mechanics was a different challenge. Quite naturally, I looked first for books written by mathematicians for mathematicians. By a stroke of bad luck, the first book I tried was to me a shining example of everything one should *not* do when writing a book. Being a mathematician does not mean that I absolutely need to hear about "a one-dimensional central extension of V by a Lie algebra 2-cocycle" just to learn the Heisenberg commutation relations. Moreover, while there is no question that mastery of some high level form of Classical Mechanics will help reaching a deeper understanding of Quantum Mechanics, Poisson Manifolds and Symplectic Geometry are not absolute prerequisites to get started. Other books written by mathematicians are less misguided, but seem to cover mostly topics which barely overlap those in physicists' textbooks with similar titles. To top it all, I was buried by the worst advice I ever received, to learn the topic from Dirac's book itself! The well-known obstacle of the difference of language and culture between mathematics and physics is all too real. To a mathematician's eye, some physics textbooks are chock-full of somewhat imprecise statements made about rather ill-defined quantities. It is not rare that these statements are simply untrue if taken at face value. Moreover arguments full of implicit assumptions are presented in the most authoritative manner. Looking at elementary textbooks can be an even

[1] Of course the realization that I could no longer do research at the level that I wanted to played a major part in this decision.

harder challenge. These often use simple-minded approaches which need not be fully correct, or try to help the reader with analogies which might be superficial and misleading.[2]

Luckily, in 2012 I ran into the preliminary version of Brian Hall's "Quantum Theory for Mathematicians" [34], which made me feel proud again for mathematicians. I learnt from this book many times faster than from any other place. The "magic recipe" for this was so obvious that it sounds trivial when you spell it out: *explain the theory in complete detail, starting from the very basics, and in a language the reader can understand*. Simple enough, but very difficult to put into practice, as it requires a lot of *humility* from the author, and humility is not the most common quality among mathematicians. I don't pretend to be able to emulate Brian's style, but I really tried and his book has had a considerable influence on mine.

After getting some (still very limited) understanding of Quantum Mechanics came the real challenge: Quantum Field Theory. I tried to learn from the most eminent[3] but this simply confirmed that I am not eminent. I then looked at books written by mathematicians for mathematicians. These were obviously not designed as first texts or for ease of reading. Moreover they focus on building rigorous theories. As of today these attempts seem to be of modest interest for most physicists.[4] More promising seemed studying Gerald Folland's heroic attempt [27] to make Quantum Field Theory accessible. His invaluable contribution is to explain what the physicists do rather than limiting the topic to the (rather small) mathematically sound part of the theory. His book is packed with an unbelievable amount of information, and, if you are stuck with minimum luggage on a desert island, this is a fantastic value. Unfortunately, as a consequence of its neutron-star density, I found it also much harder to read than I would have liked. I spent a lot of time even at places dealing with rather well-established or, worse, elementary mathematics. Sadly, this book has no real competitors and cannot be dispensed with, except by readers able to understand physics textbooks.[5] No doubt my difficulties are due to my own shortcomings, but still, it was while reading Weinberg's treatise [75] that I finally understood what induced representations are, and this is not the way it should have been. So, as the days laboring through Folland's book turned into weeks, into many

[2] I am certainly not the first mathematician to be appalled by the way physics students get treated, but it seem futile to discuss this matter further.

[3] Specifically in the book [15] by A. Connes and M. Marcolli.

[4] This being said, the muliple author treatise [10] is a magnificent piece of work. It is overwhelming at first, but most rewarding once you get into it.

[5] It is unfortunate that when this project started, the book [44] by T. Lancaster and S. Blumdell did not yet exist, for I would have saved a lot of time. While this book is certainly not designed for mathematicians, it does make a considerable effort at pedagogy, and is vastly more accessible than any other physics textbook I know of. If the reader feels like looking at a physics textbook, this should certainly be the first choice. This should also be the first choice for an introduction to the many topics we do not cover.

months, I felt the need to explain his material to myself, and to write the text from which I would have liked to learn the first steps in this area.

In the rest of the introduction I describe what I attempt to do and why. I try to provide an easily accessible introduction to some ideas of Quantum Field Theory for a reader well versed in undergraduate mathematics, but not necessarily knowing any physics beyond the high-school level or any graduate mathematics.

I must be clear about a fundamental point. A striking feature of Quantum Field theory is that it is not mathematically complete. This is what makes it so challenging for mathematicians. Numerous bright people have tried for a long time to make this topic rigorous, and have yet to fully succeed. I have nothing new to offer in this direction. This book contains statements that nobody knows how to mathematically prove. Still, I try to explain some basic facts *using mathematical language*. I acknowledge right away that familiarity with this language and the suffering I underwent to understand the present material are my only credentials for this task.

My main concern has been to spare the reader some of the difficulties from which I have very much suffered reading others' books (while of course I fear introducing new ones), and I will comment on some of these.

First, there is no doubt that the search for generality and for the "proper setting" of a theory is a source of immense progress in mathematics, but it may become a kind of disease among professional mathematicians.[6] They delight in the "second cohomology group of the Lie algebra" but do not explain why the important theorem holds when the Lie group is $\mathbb{R}$. I feel that in an introductory work generality should be indulged in only when it is useful beyond doubt. As a specific example, I see no need to mention cotangent bundles to explain basic mechanics in Euclidean space, but the use of tensor products *does* clarify Fock spaces. Rather than pursuing generality, I find more instructive to explain in complete detail simple facts and situations, especially when they are not immediately found in the literature.[7] I strive not to refer the reader to extensive specialized works, which may have different notation, and may not have been written to be easily accessible. Of course, other very different approaches are also possible [54].

Second, as an extremely ignorant person, I have suffered from the fact that many textbooks assume much known by the reader, such as "standard graduate courses in mathematics". For mathematics (except on very few occasions), I have assumed nothing that I did not learn in my first three years of university in Lyon, 1969-1972. For physics, I have tried to assume basically nothing known beyond high-school level.

Third, it was hard at first to recognize that different authors are treating in fact

[6] I personally found that the only rather accessible article of the volume [16] is due to Edward Witten! But of course this volume is not designed as an introductory course.

[7] This attitude is motivated by the fact that whatever small successes I had in my own mathematical research were always based on a thorough understanding of very simple structures.

the same material, but each in his own way. I have tried different ways to explain the material which confused me the most.

Fourth, and most importantly, I believe that brevity is not such a desirable goal that it should be reached at the reader's expense. The goal of a textbook is to communicate ideas, not to encrypt them in the shortest possible way (however beautifully this is done). Reading an introductory textbook such as this one *should simply not be a research project.* This book is long because:

- *It starts with basic material.*
- *The proofs are very detailed.*

As G. Folland appropriately points out [27], readers of notes such as these are likely to be "tourists": they do not look to acquire professional expertise in the topic. He and I disagree in that I don't think most tourists enjoy extreme sports. My overwhelming concern has been to make the book easy to read, by providing everywhere as high a level of detail as I could manage, and by avoiding anything really complicated until the last chapters. Tourists may not enjoy extreme sports, but they might enjoy leisurely sight-seeing. A number of appendices strive to provide an accessible introduction to a number of rewarding topics which complement the main story.[8]

It seemed most useful not to duplicate what is done everywhere else. First, I acknowledge the fundamental importance of historical perspective in understanding a topic, but I make no attempt whatsoever in this direction: there is no point repeating in a clumsy way what is said excellently elsewhere. Besides, one might gain by presenting early certain central and simple ideas even if they came later in the development of the theory. I am in good company here, see Weinberg's treatise [75]. Second, I concentrate on the points I had the most difficulty understanding, and I treat these in considerable detail, trying also to explain how these points are presented in physics textbooks. The subtitle of this book, a first introduction for mathematicians, *does not mean* that it intends to be the fastest possible introduction to the topic, but rather that the reader is not assumed to know anything whatsoever about it. A bare-bones treatment (covering far more material than I do) has already been written [20], and I am aiming here at a more fulfilling level of understanding. I felt it useful to cover some of the fundamental structures in sufficient detail to provide a solid foundation for the reader's future reading. One of my most glaring shortcomings is the inability to make sense of a mathematical statement unless I have taken it apart to the very last bolt and reconstructed it entirely. I have tried to do just that for these fundamental structures. This often takes several times longer than in standard textbooks. Obviously in these the reader

[8] The choice of these topics is highly personal, and reflects both my interests and the history of my learning of this topic. There are points which I felt I just *had to* understand, but which I am certain many readers will feel comfortable to accept without proof.

is expected to produce whatever efforts are required to become a professional and master the field, while I try to be much less demanding. On the other hand, some fundamentally important topics, which I found easier to learn, get only succinct coverage here when detailed understanding is not indispensable.

Quantum Field Theory is difficult and voluminous. It involves a great many deep ideas. Not all of them are extremely difficult, but the overall quantity is staggering. Choosing as I did to be simultaneously detailed and thorough limits the number of topics I might cover, and difficult choices had to be made. Roger Penrose [55], page 657, characterizes Quantum Field Theory as "this magnificent, profound, difficult, sometimes phenomenonally accurate, and yet often tantalizingly inconsistent scheme of things." As the contribution of the present work is only to describe in a self-contained manner some simple facts in mathematical language, it seemed appropriate that it covers mostly topics where mathematical language brings benefits well beyond a simple matter of notation. This is no longer the case when one ventures in the "inconsistent scheme of things" part of the theory. Still, I briefly enter this territory, in order to provide at least some form of answer to the question which serves as title, and some glimpse at physics methods. I deliberately choose to explain these as clearly as possible on the simplest possible "toy models" without any attempt to study realistic models (where the principles are similar but obscured by many accessory complications). I do not attempt what is already done in so many places, such as describing the tremendous successes of Quantum Electrodynamics. You will not learn here any real physics or what is the Standard Model. Rather, I try to prepare the reader to study books which cover these topics, among which I first recommend Folland's book [27].[9]

On the other hand, the study of renormalization, the method to circumvent the dreadful infinities, receives far more attention than it does in standard textbooks. The procedure itself is rigorous. It involves only rather elementary (but magnificently clever) mathematics. There seems to exist no other detailed source than the original papers or specialized monographs such as [47] or [61].[10] We prove in full detail the possibility of renormalization at all orders of the so-called ϕ^4_4 and ϕ^3_6 theories, cases of somewhat generic difficulty.

No magic wand will make Quantum Field Theory really easy and some effort will be required from the reader. My goal has been to make this effort easier for the reader than it was for me without sacrificing the potential for enjoyment and enrichment this fascinating topic offers.

[9] However demanding, this book is simply brilliant at a number of places, and I see no purpose to repeat the parts which I cannot improve in my own eyes.

[10] The major textbooks do not enter details in this topic, but only illustrate some ideas on examples, "referring the serious reader to the study of the original papers" see e.g. [51], p. 157. This is even the case of books which primarily deal with renormalization, such as [14], [1].

A number of people helped me while I wrote this book. Shahar Mendelson and Roy Wilsker rescued me many times from the brink of disaster. Gerald Folland spent considerable time and displayed infinite patience in trying to explain some of the most delicate points of his book [27]. Comments of Sourav Chatterjee, Carlos Guesdes, Brian Hall, Amey Joshi, Bernard Lirola, Hengrui Luo, Shuta Nakajima, Ellen Powell, Krzysztof Smuteck, Phil Sosoe [You need to reach Part II to see your name here!!] and others on the successive versions had a major impact. It was a pure delight to work with my editor Diana Gillooly.[11] I express my gratitude to them all.

[11] Unless of course the matter at hand was the length of the book.